Review of Complex Variables

The complex field consists of all solutions of polynomial equations having real coefficients. That is, the complex field is the algebraic closure of the real field.

If we denote by i and $-i$ the solutions to the equation

$$x^2 + 1 = 0,$$

then the complex field is given by

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

The complex field can be put into one-to-one correspondence with the points of a plane oriented by a rectangular coordinate system:

$$a + bi \mapsto (a, b).$$

This plane is called the complex plane. We slightly abuse notation by denoting both the complex field and the complex plane with the symbol $\mathbb{C}$.

If r and θ are the polar coordinates for an element z of the complex plane, then r is called the <u>modulus</u> and θ the <u>argument</u> of z. (In symbols: $|z| = r, \arg z = \theta$). Clearly,

$$z = r\cos\theta + i\,r\sin\theta.$$

Given $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ with $a_k, b_k \in \mathbb{R}$ the arithmetic operations on $\mathbb{C}$ are given by

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + i(b_1 + b_2) \\ z_1 z_2 &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \end{aligned}$$

The distance between z_1 and z_2 is given by

$$|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

<u>Euler's Formula</u>:

Assuming the series expansion

$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we obtain

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^z &= e^{x+iy} = e^x[\cos y + i\sin y]. \end{aligned}$$

Hence, if r and θ are the polar coordinates for $z = x + iy$, then $z = r\,e^{i\theta}$. Moreover, we immediately obtain De Moivre's Formula

$$\begin{aligned} z_1 z_2 \cdots z_n &= (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) \cdots (r_n e^{i\theta_n}) \\ &= r_1 r_2 \cdots r_n e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \\ &= r_1 \cdots r_n(\cos(\theta_1 + \cdots + \theta_n) + i\sin(\theta_1 + \cdots + \theta_n)) \end{aligned}$$

where $z_k = r_k e^{i\theta_k}$, $k = 1, \ldots, n$. We also obtain

$$z^n = (r\,e^{i\theta})^n = r^n e^{in\theta}$$

and

$$\sqrt[n]{z} = z^{1/n} = r^{1/n} e^{i\theta/n}.$$

In particular, the numbers $1, w, w^2, \ldots, w^{n-1}$ where $w = e^{2\pi i/n}$ are called the n^{th} roots of unity. They consistute the n distinct roots of the polynomial equation

$$x^n = 1.$$

The complex conjugate of a point $z = a + bi$, a, $b \in \mathbb{R}$, is the point $\overline{z} = a - bi$. Observe that

$$z\overline{z} = |z|^2$$

and if $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$. Given $z_0 \in \mathbb{C}$ and $\epsilon > 0$, we denote by

$$B(z_0, \epsilon) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

the open disk (or ball) of radius ϵ about z_0.

Point Set Terminology

- A set $S \subset \mathbb{C}$ is said to be open if for each point $z \in S$ there is an $\epsilon > 0$ such that

$$B(z, \epsilon) \subset S.$$

- The complement of a set $S \subset \mathbb{C}$ is given by

$$\widetilde{S} = \mathbb{C} \backslash S := \{z \in \mathbb{C} : z \notin S\}.$$

- A set $S \subset \mathbb{C}$ is said to be closed if $\widetilde{S}$ is open.
- A point $z \in \mathbb{C}$ is said to be a boundary point of the set $S \subset \mathbb{C}$ if all $\epsilon > 0$ the set $B(z, \epsilon)$ contains points from both S and $\widetilde{S}$. The collection of all boundary points of S is called the boundary of S and is denoted bdry S.
- Any two continuous functions $x(t)$ and $y(t)$ parametrized on the real interval $[\alpha, \beta] \subset \mathbb{R}$ generate a continuous curve in $\mathbb{C}$ given by

$$z(t) := x(t) + iy(t), t \in [\alpha, \beta].$$

  Such a curve is called a Jordan Arc if it contains no multiple points, i.e., distinct points on the curve correspond to distinct values of t. Such curves are oriented in the sense that there is an initial point $z(\alpha)$, a terminal point $z(\beta)$, and for any two points on the curve it is always clear which precedes the other.
- A closed Jordan curve (simple closed curve) is a continuous curve having $z(\alpha) = z(\beta)$ but otherwise having no other multiple points.
- A Jordan arc need not be of finite arc length. But if it is, then the arc is said to be rectifiable and is then called a path segment.
- If a finite number of path segments are joined in order in such a manner that the initial point of each coincides with the terminal point of its predecessor, a path is formed. If the initial and terminal points of a path coincide, it is called a closed path. A closed path is oriented as before in the sense that $z(t)$ describes the entire closed path precisely once when t runs over its interval. If two distinct points z always correspond in this manner to two distinct values of t, except the initial and terminal values, the closed path is said to be simple.
- A set S is said to be connected if any two points of the set can be joined by a path consisting of straight line segments (i.e. polygonal path) all points of which are in S.
- An open connected set is called an (open) region or domain.
- The closure of a set is the union of the set with its boundary points. A closed region (or domain) is a set that is the closure of an open region (or domain).
- A region R is called simply-connected if any closed Jordan curve in R can be continuously shrunk to a point without leaving R. A region that is not simply-connected is called multiply-connected.

Jordan Curve Theorem:

A closed Jordan curve decomposes the plane into precisely two separate regions one of which is bounded and the other unbounded. The curve forms the common boundary of both regions.

The bounded region is called the interior or inside of the curve and the unbounded region is called the exterior or outside of the curve.

It follows that the region inside a simple closed curve is simply-connected.

Functions of a Complex Variable

Let $S \subset \mathbb{C}$. A function from S to $\mathbb{C}$ is a correspondence between points in S and points in $\mathbb{C}$. Each $z \in S$ may correspond to one or more values w in $\mathbb{C}$. We say that w is a function of z and write $w = f(z)$ or $w = G(z)$ depending on the name that is assigned to the correspondence. A function that assigns one and only one value of $w \in \mathbb{C}$ to each value $z \in S$ is said to be single-valued on S; otherwise, it is said to be multi-valued.

ex: 1) $f(z) = z^2$ is single-valued

2) $f(z) = z^{1/2}$ is multi-valued.

$$\begin{aligned} z &= re^{i(\theta+2k\pi)} \qquad && k = 0, 1, 2, \ldots \\ z^{1/2} &= r^{1/2}e^{i(\frac{\theta}{2}+k\pi)} \qquad && k = 0, 1, \ldots \\ &= \pm r^{1/2}e^{i\theta/2} \end{aligned}$$

A multi-valued function can often be considered as a collection of single-valued functions, each member of which is called a branch of the function. In such cases it is customary to consider one particular member as the principle branch of the multi-valued function.

- A single-valued function $f : \mathbb{C} \to \mathbb{C}$ is said to be continuous at the point $z_0 \in \mathbb{C}$ if to every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \epsilon$.
- The function f is said to be continuous on a set S if it is continuous at every point of S.
- Just as in the case of real variables, closed and bounded subsets of $\mathbb{C}$ are compact, i.e., bounded infinite sequences admit at least one convergent subsequence. Moreover, continuous functions on compact sets are uniformly continuous.
- Recall that each complex number can be associated with a unique pair of real numbers

$$z = x + iy \mapsto (x, y) \in \mathbb{R}^2$$

where x is called the real part of z, $\mathrm{Re}(z)$, and y the imaginary part of z, $\mathrm{Im}(z)$. In this way, each function $f : \mathbb{C} \to \mathbb{C}$ corresponds to a pair of mappings $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x + iy) = u(x, y) + iv(x, y).$$

The function u is called the real part of f and v the imaginary part. We have that f is continuous if and only if both u and v are continuous.

- If $f : \mathbb{C} \to \mathbb{C}$ is single-valued in some domain $D \subset \mathbb{C}$, the derivative of f at a point z is defined as

$$f'(z) := \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided the limit exists and is independent of the manner in which $\Delta z \to 0$. In such cases $f(z)$ is said to be differentiable at z.

- If $f'(z)$ exists at all points z in a domain D, then $f(z)$ is said to be analytic in D. The terms regular and holomorphic are sometimes used as synonyms for analytic.

ex: 1) $f(z) = z^\alpha$, $f'(z) = \alpha z^{\alpha - 1}$

2) $f(z) = e^z, f'(z) = e^z$

Cauchy-Riemann Equations:

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D is that u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{C-R-E}$$

If these partial derivatives are continuous in D, then the Cauchy-Riemann equations are sufficient for f to be analytic in D.

Proof: (Necessity) Let $z + \Delta z = (x + \Delta x) + i(y + \Delta y)$. In order that f be analytic, the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(x)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + \Delta x + i(y + \Delta y)) - f(x + iy)}{\Delta x + i\Delta y}$$

must exist and be independent of the manner in which $\Delta x \to 0$, $\Delta y \to 0$. We consider the limit on two trajectories having $\Delta x \to 0$, $\Delta y \to 0$.

(1) $\Delta y = 0$, $\Delta x \to 0$: Then

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \left[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - V(x, y)}{\Delta x} \right] \\ &= \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y) \end{aligned}$$

(2) $\Delta x = 0$, $\Delta y \to 0$: Then

$$\begin{aligned} f'(z) &= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x,y+\Delta y) - v(x,y)}{i\Delta y} \right] \\ &= -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y). \end{aligned}$$

By equating the real and imaginary parts in (1) and (2) above, we obtain the Cauchy-Riemann equations.

(Sufficiency) From the continuity of the partials in C-R-E, we know that

$$\begin{aligned} f(x+\Delta x,y+\Delta y)-f(x,y) &= u(x+\Delta x,y+\Delta y)-u(x,y)+i(v(x+\Delta x,y+\Delta y)-v(x,y)) \\ &= [(u(x+\Delta x,y+\Delta y)-u(x,y+\Delta y))+(u(x,y+\Delta y)-u(x,y))] \\ &\quad +i[(v(x+\Delta x,y+\Delta y)-v(x,y+\Delta y))+(v(x,y+\Delta y)-v(x,y))] \\ &= \tfrac{\partial u}{\partial x}\Delta x+0(\Delta x)+\tfrac{\partial u}{\partial y}\Delta y+0(\Delta y) \\ &\quad +i\left[\tfrac{\partial v}{\partial x}\Delta x+0(\partial x)+\tfrac{\partial v}{\partial y}\Delta y+0(\Delta y)\right] \\ &= \left[\tfrac{\partial u}{\partial x}+i\tfrac{\partial v}{\partial x}\right]\Delta x+\left[\tfrac{\partial u}{\partial y}+i\tfrac{\partial v}{\partial y}\right]\Delta y+0(\Delta x)+0(\Delta y) \\ \text{(CRE)} &= \left[\tfrac{\partial u}{\partial x}+i\tfrac{\partial v}{\partial x}\right]\Delta x+\left[-\tfrac{\partial v}{\partial x}+i\tfrac{\partial u}{\partial x}\right]\Delta y+0(\Delta x)+0(\Delta y) \\ &= \left(\tfrac{\partial u}{\partial x}+i\tfrac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)+0(\Delta x)+0(\Delta y) \end{aligned}$$

Now taking the limit as $\Delta z = \Delta x + i\Delta y \to 0$ we find that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

so that the limit exists and is unique. $\square$

<u>Singular Points</u>:

A point at which f fails to be analytic is called a singular point or singularity of f.

Types of singularities:

1) <u>Isolated singularities</u>: A point z_0 is called an isolated singularity of f if there is an $\epsilon > 0$ such that the only singularity of f in the ball $B(z_0; \epsilon)$ is z_0.
2) <u>Poles</u>: A point z_0 is said to be a pole of f of order $n \in \{1, 2, \ldots\}$ if

$$\lim_{z \to z_0} (z - z_0)^n f(z)$$

exists and is non-zero. If $n = 1$, the pole is said to be simple.

3) Branch points of multi-valued functions are singularities.

4) Removable singularities: A singularity at which

$$\lim_{z \to z_0} f(z) = L$$

exists but at which $f(z)$ is either undefined or $f(z_0) \neq L$ is called a removable singularity.

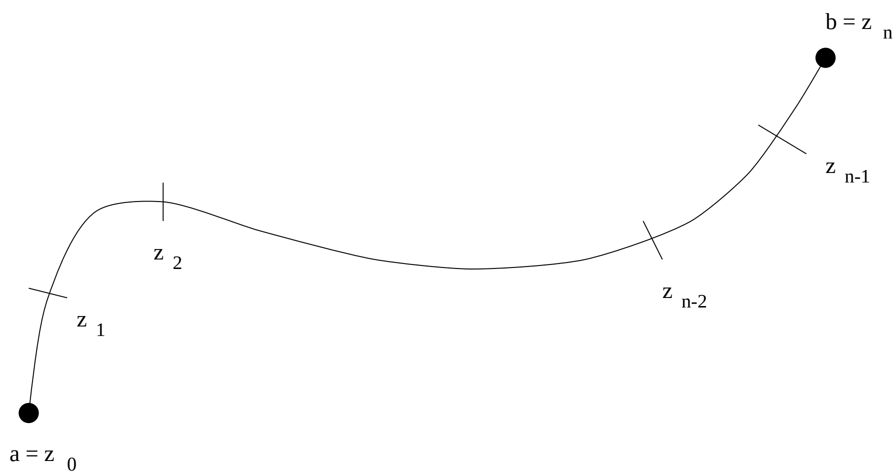
<u>ex</u>: $f(z) = \frac{\sin z}{z}$, $z = 0$

5) Essential singularity: A singularity that is neither a pole, a branch point, or removable is called an essential singularity.

<u>Complex Integration</u>

<u>Complex line integrals</u>:

Let f be continuous at all points of a continuous rectifiable curve C. Partition C into n parts by means of points $z_1, \ldots, z_{n-1}$ chosen arbitrarily and set $z_0 = a$, $z_n = b$



One each of the arcs from z_{k-1} to z_k choose a point ζ_k for $k = 1, \ldots, n$. Form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\zeta_k)\Delta z_k.$$

If the limit of the S_n's exists as $\max_k \Delta z_k \to 0$ independent of the choice of partition and the selection of the points ζ_k, then we say that the complex line integral along C from a to b exists and we say that the value of the integral, denoted $\int_C f(z)dz$, is equal to the value of the limit. Note that if f is bounded on C by M, i.e.

$$|f(z)| < M \qquad \text{for all } z \in C,$$

then

$$\left|\int_C f(z)dz\right| \le M \cdot [\text{arclength of } C].$$

Real Line Integrals:

Recall that real line integrals are defined in the same way via Riemann sums. If P and Q are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ and C is a smooth curve in $\mathbb{R}^n$ parametrized by the differentiable functions φ and ψ;

$$C = \{(\varphi(t), \psi(t)) : t \in [a, b]\}.$$

Then

$$\begin{aligned}&\int_C [P(x,y)dx + Q(x,y)dy] \\ &\quad = \int_a^b [P(\varphi(t), \psi(t))\varphi'(t)dt + Q(\varphi(t), \psi(t))\psi'(t)dt]\end{aligned}$$

If $f(x+iy) = u(x,y) + iv(x,y)$ and C is a rectifiable curve for which $\int_C f dz$ exist, then one can show by real variable techniques that

$$\begin{aligned}\int_C f(z)dz &= \int_C (u+iv)(dx+idy) \\ &= \int_C (udx - vdy) + i\int_C (vdx + udy).\end{aligned}$$

Thus the real and imaginary parts of $\int_C f dz$ can be computed by real variable methods.

Contour integrals:

Let C be the boundary of a region of the (x,y)-plane. Let an observer be standing on that side of the plane such that if the observer were at the origin and looking down the positive x-axis, then the positive y-axis would be on their left, i.e., the top of their head points in the direction of the positive z-axis in right-handed three dimensional coordinates. The curve C is said to be traversed in the positive direction if the observer, traveling in this direction, has the region to the left. We use the symbol

$$\oint_C f(z)dz$$

to denote the integration of f around the boundary of C in the positive direction. In the case of a circle, the positive direction is counterclockwise. Such an integral around C is called a contour integral.

Green's Theorem in the Plane:

Let $P(x,y)$ and $Q(x,y)$ be continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ having continuous partial derivatives in the region R and its boundary. Then

$$\oint_C Pdx + Qdy = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

The theorem is valid for both simply and multiply connected regions.

Cauchy's Theorem:

Let f be analytic in a region R and on its boundary, C, if f' is continuous on both R and C. Then

$$\oint_C f(z)dz = 0$$

Proof: $\oint_C f(z)dz = \oint_C (u+iv)(dx+idy) = \oint_C (udx - vdy) + i(vdx + udy)$
$= \iint\limits_R -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$ (by Green's Theorem)
$= 0.$ (by the CRE) $\square$

Note: The so-called Cauchy-Goursat Theorem does not require f' to be continuous on R and C to obtain the result. But the proof is substantially more intricate.

Corollary 1: (Path Independence). *If f is analytic in a simply connected region R and z_0 and z_1 are any two points in R, the*

$$\int_{C_1} fdz = \int_{C_2} fdz$$

for any two paths connecting z_1 to z_2.

Corollary 2: (Existence of Antiderivatives). *Let f be analytic on a simply connected region R and let $z_0 \in R$. For each $z \in R$ define*

$$F(z) = \int_{z_0}^{z} f(\xi)d\xi.$$

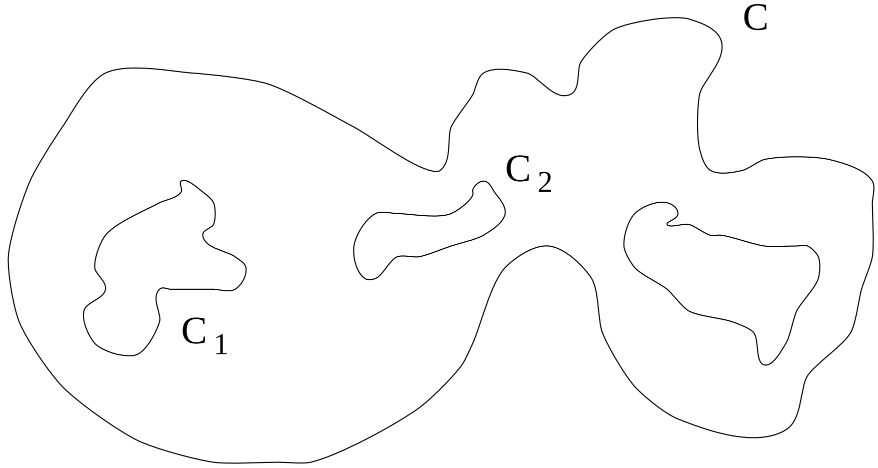
Then F is analytic on R with $F'(z) = f(z)$.

Corollary 3: (Fundamental Theorem of Calculus). *Let F have an analytic derivative $f = F'$ on the simply connected region R. Then*

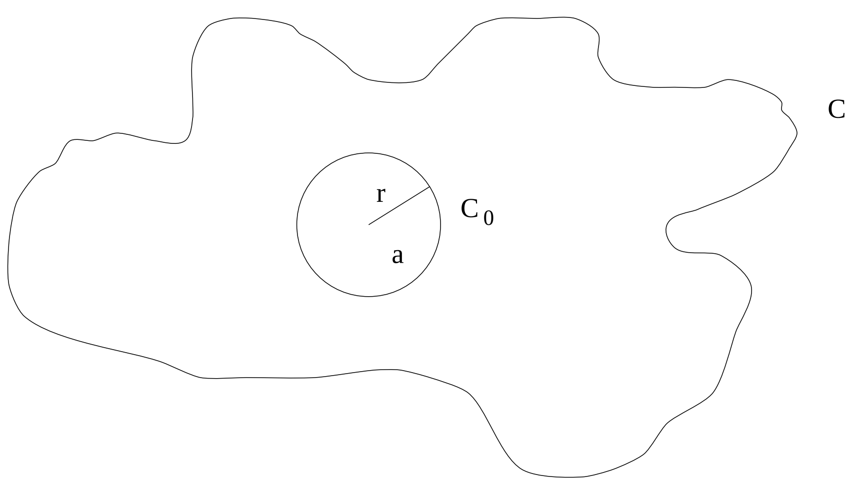
$$\int_a^b f(z)dz = F(b) - F(a).$$

Corollary 4: (Homotopic Equivalence of Integrals). *Let f be analytic in a region bounded by non-overlapping simple closed curves C, C_1, $C_2, \ldots, C_n$ where $C_1, C_2, \ldots, C_n$ are contained in the interior of the bounded region determined by C. If f is also analytic on $C, C_1, \ldots, C_n$, then*

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz.$$



<u>example</u>: Evaluation $\oint_C \frac{dz}{z-a}$ on every simple closed curve bounding a region containing a in its interior.



$$\begin{aligned}\oint_C \frac{dz}{z-a} &= \oint_{C_0} \frac{dz}{z-a}, |z-a| = r, z = a + re^{i\theta}, 0 \le \theta \le 2\pi \\ &= \int_n^{2\pi} \frac{rie^{i\theta}d\theta}{re^{i\theta}} = 2\pi i\end{aligned}$$

Leibniz's Rule

Let $\varphi : [a, b] \times [c, d] \to \mathbb{C}$ be a continuous function and define $g : [c, d] \to \mathbb{C}$ by

$$g(t) = \int_a^b \varphi(s, t) ds.$$

Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a, b] \times [c, d]$, then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) ds. \tag{LR}$$

Proof: The continuity of g follows from the uniform continuity of φ on $[a, b] \times [c, d]$. If we now prove that g is differentiable with g' given by formula (LR), then it will follow from the first part that g' is continuous since $\frac{\partial \varphi}{\partial t}$ is continuous. Hence, we need only verify the formula (LR).

Fix a point t_0 in $[c, d]$ and let $\epsilon > 0$. Denote $\partial \varphi / \partial t$ by φ_2: it follows that φ_2 is uniformly continuous on $[a, b] \times [c, d]$. Thus, there is a $\delta > 0$ such that $|\varphi_2(s', t') - \varphi_2(s, t)| < \epsilon$ whenever $(s - s')^2 + (t - t')^2 < \delta^2$. In particular,

$$|\varphi_2(s, t) - \varphi_2(s, t_0)| < \epsilon$$

whenever $|t - t_0| < \delta$ and $a \leq s \leq b$. Hence

$$\left| \int_{t_0}^t [\varphi_2(s, \tau) - \varphi_2(s, t_0)] d\tau \right| \leq \epsilon |t - t_0| \tag{a}$$

whenever $|t - t_0| < \delta$ and $a \leq s \leq b$. Now for a fixed s in $[a, b]$ the function

$$\Phi(t) = \varphi(s, t) - t\varphi_2(s, t_0)$$

is an anti-derivative for $\Psi(t) = \varphi_2(s, t) - \varphi_2(s, t_0)$. Thus, by the Fundamental Theorem of Calculus with (a), it follows that

$$|\varphi(s, t) - \varphi(s, t_0) - (t - t_0)\varphi_2(s, t_0)| \leq \epsilon |t - t_0|$$

for any s when $|t - t_0| < \delta$. Therefore,

$$\begin{aligned} &\left| \frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \varphi_2(s, t_0) ds \right| \\ &\leq \int_a^b \left| \frac{\varphi(s, t) - \varphi(s, t_0) - (t - t_0)\varphi_2(s, t_0)}{t - t_0} \right| ds \\ &\leq \epsilon |b - a| \end{aligned}$$

whenever $|t - t_0| < \delta$. $\square$

Corollary. *Let $G \subset \mathbb{C}$ be open and let C be a rectifiable curve (path) in G. Suppose that $\varphi : C \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by*

$$g(z) = \int_C \varphi(\zeta, z) d\zeta.$$

Then g is continuous. If $\frac{\partial \varphi}{\partial z}$ exists for each (ζ, z) in $C \times G$ and is continuous, then g is analytic and

$$g'(z) = \int_C \frac{\partial \varphi}{\partial z}(\zeta, z) d\zeta.$$

Cauchy Integral Formulas

If f is analytic inside and on a simple closed curve C and z is any point inside C, then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Moreover, the n^{th} derivative of f at z is given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^n} d\zeta \tag{CIF}$$

for $n = 0, 1, 2, \ldots$.

Proof. We proceed by induction on n in (CIF). Let $n = 0$. First note that by Cor. 4 to Cauchy's Theorem we may replace C by the curve Γ_ϵ given by $w = z + \epsilon e^{i\theta}$, $0 \le \theta \le 2\pi$ for any given $\epsilon > 0$. Next observe that

$$\begin{aligned}
\oint_{\Gamma_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta &= \oint_{\Gamma_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + \oint_{\Gamma_\epsilon} \frac{f(z)}{\zeta - z} d\zeta \\
&= \oint_{\Gamma_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + 2\pi i f(z)
\end{aligned}$$

by the example appearing after the above mentioned corollary. Thus, it remains only to show that

$$\oint_{\Gamma_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = 0.$$

But, by the definition of the derivative,

$$\left| \frac{f(\zeta) - f(z)}{\zeta - z} - f'(z) \right| \le o(\epsilon) \text{ on } \Gamma_\epsilon.$$

Letting $\epsilon \to 0$ yields the result for $n = 0$.

Now suppose that (CIF) holds for $n = 0, 1, \ldots, N$. We show that it must also hold at $N+1$. For this we apply the Corollary to Leibniz's Rule to the formula (CIF) at $n = N$:

$$\begin{aligned} f^{(N+1)}(z) &= \frac{\partial}{\partial z} f^{(N)}(z) \\ &= \frac{N!}{2\pi i} \frac{\partial}{\partial z} \oint_C \frac{f(\zeta)}{(\zeta - z)^{N+1}} d\zeta \\ &= \frac{N!}{2\pi i} \oint_C f(\zeta) \left[\frac{\partial}{\partial z} (\zeta - z)^{-(N+1)} \right] d\zeta \qquad \text{(by Liebniz's rule)} \\ &= \frac{(N+1)!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{N+2}} d\zeta. \end{aligned}$$

□

Corollary 1: (Morera's Theorem) (Converse of Cauchy's Theorem). *If f is continuous in a simply-connected region $\mathbb{R}$ and if*

$$\oint_C f(z) dz = 0$$

around every simple closed curve in R, then f is analytic in R.

Proof: If $\oint_C f(z) dz = 0$ independent of C, then the function

$$F(z) = \int_{z_0}^{z} f(\zeta) d\zeta$$

is well defined and $F'(z) = f(z)$. Hence F is analytic on R. The Cauchy Integral Formula's imply F'' exists and is continuous on R. But $F'' = f'$ so f is analytic on R. □

Corollary 2: (Cauchy's Inequality). *If f is analytic inside and on a circle C of radius r centered at z_0, then*

$$|f^{(n)}(z_0)| \le \frac{M \cdot n!}{r^n}, n = 0, 1, 2, \ldots$$

where M is any constant such that $|f(z)| < M$ for all z on C.

Proof: By CIF,

$$\begin{aligned} |f^{(n)}(z_0)| &= \frac{n!}{2\pi} \left| \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| d\zeta \\ &\le \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r. \end{aligned}$$

□

Corollary 3: (Liouville's Theorem). *The only functions that are both analytic and bounded on all of $\mathbb{C}$ are the constant functions.*

Proof: Assume f is analytic and bounded on $\mathbb{C}$. We need to show that $f' \equiv 0$. Let $z_0 \in \mathbb{C}$, $r > 0$, and $M > 0$ be a bound for f on $\mathbb{C}$. Then, by Cauchy's Inequality,

$$|f'(z_0)| \leq \frac{M}{r}.$$

Send $r \to \infty$ to obtain the result. $\square$

Corollary 4: (The Fundament Theorem of Algebra). *Every polynomial equation*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n = 0,$$

where the degree $n \geq 1$ and $a_n \neq 0$, has exactly n roots (counting multiplicity).

Proof: We first show that P has at least one root. Indeed, if P had no roots, then $q(z) = 1/P(z)$ would be analytic on $\mathbb{C}$. Moreover, $|q(z)|$ is bounded, in fact approaches zero, as $|z| \to \infty$. Thus, by Liouville's Theorem, $q(z)$ is the constant function. This contradicts the hypothesis that $n \geq 1$ so P has at least one root.

Let z_1 be a root of P. Then by the division algorithm for polynomials

$$P(z) = (z - z_1)P_1(z)$$

where P_1 is a polynomial of degree $n - 1$. Hence P_1 has a root z_2. Thus,

$$P(z) = (z - z_1)(z - z_2)P_2(z)$$

where P_2 is a polynomial of degree $n - 2$. Continue in this way to obtain the result. $\square$

Power Series, Taylor Series, and Laurent Series

Let $u_1(z), u_2(z), \ldots$, be a sequence of functions defined and single-valued in some region $R \subset \mathbb{R}$. We call $u(z)$ the (pointwise) limit of the sequence $\{u_n\}$ on R, and write $\lim_n u_n = u$, if for each $\epsilon > 0$ and $z \in R$ there is an $N_{(\epsilon,z)}$ such that

$$|u_n(z) - u(z)| < \epsilon \quad \forall\, n \geq N_{(\epsilon,z)}.$$

In this case, the sequence $\{u_n\}$ is said to converge (pointwise) to u on R. If it so happens that for each $\epsilon > 0$ there is an $N(\epsilon)$ such that

$$|u_n(z) - u(z)| < \epsilon \quad \forall\, n \geq N_{(\epsilon)} \text{ and } z \in R,$$

we say that the sequence $\{u_n\}$ converges uniformly to u on R.

One can associate with the sequence of functions $\{u_n\}$ a new sequence of functions

$$\begin{aligned} s_1(z) &= u_1(z) \\ s_2(z) &= u_1(z) + u_2(z) \\ &\vdots \\ s_n(z) &= u_1(z) + u_2(z) + \cdots + u_n(z) \\ &\vdots \end{aligned}$$

called the sequence of partial sums with s_n called the n^{th} partial sum.

The sequence $\{s_n\}$ is symbolized by the notation

$$\sum_{n=1}^{\infty} u_n$$

and is called an infinite series. If $\lim_n s_n$ exists, the series is said to be convergent; otherwise, it is said to be divergent. The series is called absolutely convergent if the associated series of absolute values

$$\sum_{n=1}^{\infty} |u_n(z)|$$

converges. If $\sum_{n=1}^{\infty} u_n(z)$ converges but $\sum_{n=1}^{\infty} |u_n(z)|$ does not, the series is said to be conditionally convergent. The series $\sum_{n=1}^{\infty} u_n(z)$ is said to be uniformly convergent on a region R if the associated sequence of partial sums is uniformly convergent on R.

A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a power series. The radius of convergence of a power series is given by

$$R = \sup\{r \geq 0 : \sum_{n=0}^{\infty} a_n r^n \text{ converges}\},$$

and the disk $\{z : |z - z_0| \leq R\}$ is called the circle of convergence. The radius of convergence may take the values 0 and $+\infty$ and it always exists ($R = 0$ is always possible).

Theorem. *If R is the radius of convergence for the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then the series converges uniformly and absolutely on any bounded region whose boundary lies entirely inside the circle of convergence, the series diverges at every point outside the circle of convergence, and the series can either diverge or converge at boundary points of the circle of convergence.*

Theorem: [Term by Term Integration and Differentiation]. *Let $f_0, f_1, \ldots$ be a sequence of functions each of which is analytic on the region R. Further suppose that the series*

$$F(z) = \sum_{n=0}^{\infty} f_n(z)$$

is uniformly convergent on every closed subregion of R.

(a) The function $F(z)$ is analytic on R with

$$F^{(k)}(z) = \sum_{n=0}^{\infty} f_n^{(k)}(z) \qquad k = 1, 2, \ldots$$

and these series converge uniformly on every closed subregion of R. In particular, a power series can be differentiated term by term in any region lying entirely inside its circle of convergence.

(b) Every series obtained by integration term by term along a path C in R converges and

$$\int_C F(z)dz = \sum_{n=0}^{\infty} \int_C f_n(z)dz.$$

In particular, a power series can be integrated term by term along any curve lying entirely inside its circle of convergence.

Geometric Series:

The geometric series $\sum_{n=0}^{\infty} r^n$ is perhaps the most important key available in the study of infinite series and power series in particular. We make use of the following facts:

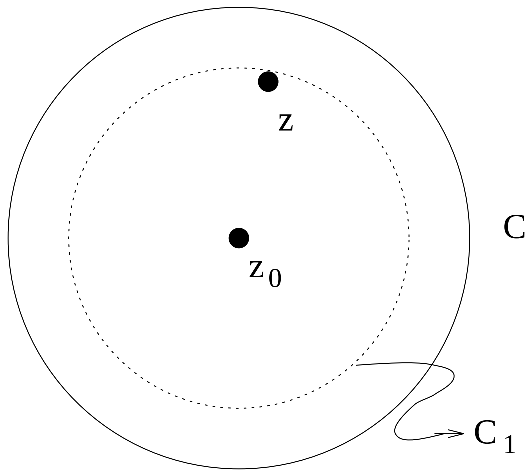
1) $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$ for all $r \in \mathbb{C}$.
2) $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if and only if $|r| < 1$.
3) $\frac{1}{1-r} = 1 + r + r^2 + \cdots + r^{n-1} + \frac{r^n}{1-r}$ for all $r \in \mathbb{C}$.

Note: (3) is just a restatement of (1).

Taylor's Theorem. *If f is analytic inside a circle C with center z_1, then for all z inside C,*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f'(z_0)}{2!}(z - z_0)^2 + \cdots$$

Proof: Let z be any point inside C. Construct a circle C_1 with center at z_0 and enclosing z.



Then by Cauchy's integral formula,

$$(a) \qquad f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Let $r = (z - z_0)/(\zeta - z_0)$, then

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\frac{1}{1 - r} \\
&= \frac{1}{\zeta - z_0}\left[1 + r + r^2 + \cdots + r^{n-1} + \frac{r^n}{1 - r}\right] \\
&= \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \cdots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \left(\frac{z - z_0}{(\zeta - z_0}\right)^n \frac{1}{\zeta - z}.
\end{aligned}$$

By substituting this identity into (a) above we obtain

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z_0} d\zeta + \frac{(z - z_0)}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta + \cdots$$

$$+ \frac{(z - z_0)^{n-1}}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^n} d\zeta + E_n$$

where $E_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \frac{f(\zeta)}{\zeta-z}d\zeta$. Applying the Cauchy Integral Formulas yields

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + E_n.$$

Hence, it remains only to show that $E_n \to 0$ as $n \to \infty$. First note that since ζ is on C, we have

$$\left|\frac{z-z_0}{\zeta-z_0}\right| = \gamma < 1$$

where γ is constant for all ζ on C_1. Also, $|f(\zeta)| < M$ for some $M > 0$ on C_1, and $|\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq P - |z - z_0| > 0$ where P is the radius of C_1. Therefore,

$$\begin{aligned} |E_n| &= \frac{1}{2\pi}\left|\oint_{C_1}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \frac{f(\zeta)}{\zeta-z}d\zeta\right| \\ &= \frac{1}{2\pi}\frac{\gamma^n M}{P-|z-z_0|}2\pi P \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

□

<u>Some Special Series</u>:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \cdots \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \\ \ln(1+z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \end{aligned}$$

Laurent's Theorem. *If f is analytic inside and on the boundary of the annular shaped region R bounded by two concentric circles C_1 and C_2 with center at z_0 and respective radii r_1 and r_2 $(r_1 > r_2)$, then for all z in R,*

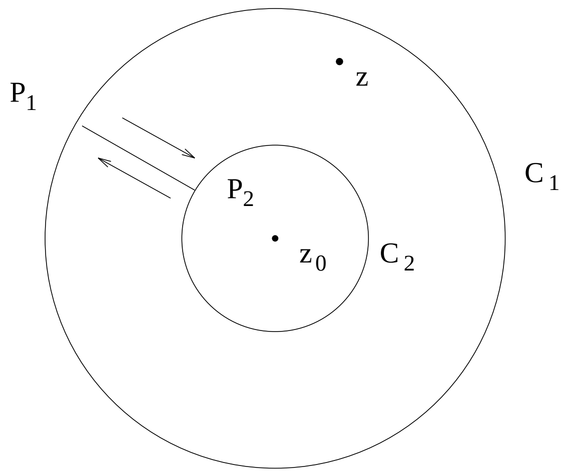
(LS) $$f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=1}^{\infty}\frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta, n = 0, 1, 2, \ldots \\ a_{-n} &= \frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{(\zeta-a)^{-n+1}}d\zeta, n = 1, 2, \ldots \end{aligned}$$

Proof: Let $z \in R$. Observe that by Cauchy's Integral formula

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z)} d\zeta$$



The path integrals between
P_1 and P_2 cancel each
other out

Let us first consider the integral on C_1. Observe that if ζ is a point on C_1, then we have the expansion

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{1}{1 - \left(\frac{z-z_0}{\zeta - z_0}\right)} = \frac{1}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta - z_0}\right)^n$$

where this series converges uniformy for all ζ on C_1 since

$$\left|\frac{z-z_0}{\zeta - z_0}\right| = \frac{|z-z_0|}{r_1} < 1.$$

Hence

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta &= \frac{1}{2\pi i}\oint_{C_1} \sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z-z_0)^n d\zeta \\ &= \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{(by uniform convergence).}\end{aligned}$$

Next consider the integral $\frac{-1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{\zeta - z}d\zeta$. If ζ is a point on C_2, we have the expansion

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0}\frac{1}{1 - \left(\frac{\zeta-z_0}{z - z_0}\right)} = -\frac{1}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{\zeta-z_0}{z - z_0}\right)^n$$

where this series converges uniformly for all $\zeta \in C_2$ since

$$\left|\frac{\zeta - z_0}{z - z_0}\right| = \frac{r_2}{|z - z_0|} < 1.$$

Hence

$$\begin{aligned}
-\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z)} d\zeta &= \frac{1}{2\pi i}\oint_{C_2} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} f(\zeta)(\zeta - z_0)^n d\zeta \\
&= \sum_{n=0}^{\infty} a_{-1}(z - z_0)^n \quad \text{(by uniform convergence).} \qquad \square
\end{aligned}$$

The first term in (LS) is called the principle part and the second the analytic part of the Laurent series.

Classification of Singularities

It is possible to classify the singularities of a function f by examination of its Laurent series expanded at these singularities. First note that if f is analytic inside the circle C_2, then by Cauchy's Theorem $a_{-n} = 0$ for all $n = 1, 2, \ldots$, and the Laurent series reduces to the Taylor series for the function.

1) Poles: If $f(z)$ has expansion (LS) in which the principle part has only finitely many terms given by

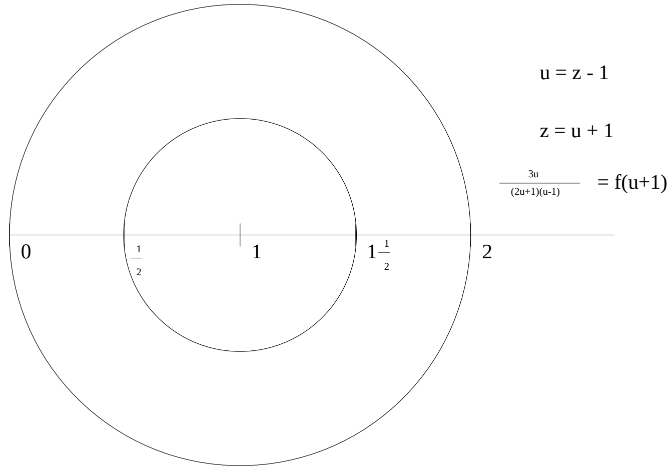
$$\frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-n}}{(z - z_0)^n}$$

where $a_{-n} \neq 0$, then z_0 is a pole of order n for f. If $n = 1$, then it is a simple pole.

2) Essential singularities: If $f(z)$ has expansion (LS) in which the principle part has infintely many terms, then z_0 is an essential sinsgularity for f.

A function is said to be entire if it is analytic on all of $\mathbb{C}$. If f is analytic on a region R except for finitely many poles, then f is said to be meromorphic on R. One rarely computes Laurent series by evaluating the integrals in (LS), rather one builds the series from known series just as in the case of Taylor Series. Some examples follow:

examples: 1) $f(z) = \frac{3z-3}{(2z-1)(z-2)}$; Find a Laurent series for f about $z = 1$ convergent for $\frac{1}{2} < |z - 1| < 1$.



<u>Partial Fractions</u>

$$\frac{3u}{(2u+1)(u-1)} = \frac{1}{2u+1} + \frac{1}{u-1}, \frac{1}{2} < |u| < 1$$

$$\frac{1}{u-1} = -\sum_{n=0}^{\infty} u^n = -\sum_{n=0}^{\infty} (z-1)^n, |u| < 1$$

$$\frac{1}{2u+1} = \frac{1}{2u}\frac{1}{1+\frac{1}{2u}} = \frac{1}{2u}\sum_{n=0}^{\infty}(-1)^n(2u)^{-n}, \left|\frac{1}{2u}\right| < 1, \frac{1}{2} < |u|$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{n+1}}\frac{1}{u^{n+1}} = -\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\left(\frac{1}{z-1}\right)^n.$$

Why is the principle part of this expansion infinite?

2) $$f(x) = \frac{3z-3}{(2z-1)(z-2)} \text{ at } z = 2, u = z-2, z = u+2$$

$$= \frac{3u+3}{(2u+3)u} = \left(1+\frac{1}{u}\right)\frac{1}{1+\frac{2}{3}u}$$

$$= \left(1+\tfrac{1}{u}\right)\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n u^n, |u| \le \frac{3}{2}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n u^n + \sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n u^{n-1}$$

$$= \frac{1}{u} + \sum_{n=0}^{\infty}\left[\left(-\frac{2}{3}\right)^n + \left(-\frac{2}{3}\right)^{n+1}\right]u^n$$

$$= \frac{1}{z-2} + \sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n\left[1-\frac{2}{3}\right](z-2)^n$$

$$= \frac{1}{z-2} + \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n (z-2)^n$$

<u>Integration around singularities</u>:

1) It has already been shown that

$$\oint_C \frac{dz}{z-z_0} = 2\pi i, \text{ where } C = \{z : z = z_0 + e^{i\theta}, 0 \le \theta \le 2\pi\}$$

2) Evaluate the integral $\oint_C \frac{dz}{(z-z_0)^n}$, $n = 2, 3, \ldots$ on any simple closed path having z_0 in the interior of the bounded region determined by C.

First observe that by Cauchy's Theorem we may as well take $C = \{z : z = z_0 + e^{i\theta}, 0 \le \theta \le 2\pi\}$. Then

$$\begin{aligned}
\oint_C \frac{dz}{(z-z_0)^n} &= \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{in\theta}} & \begin{array}{l} z = z_0 + e^{i\theta} \\ dz = ie^{i\theta}d\theta \end{array} \\
&= i\int_0^{2\pi} e^{i(1-n)\theta}d\theta \\
&= \frac{1}{1-n}e^{i(1-n)\theta} \Big|_0^{2\pi} = 0, n = 2, 3, \ldots
\end{aligned}$$

Let C be a simple closed curve and suppose that f is single-valued and analytic inside and on C except at the point z_0 lying inside C. If, at z_0, f has the Laurent expansion

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n,$$

then $\oint_C f(z)dz = 2\pi i a_{-1}$. For this reason, a_{-1} is called the residue of f at z_0.

In order to see this, first use Cauchy's Theorem to shrink C to a circle Γ of radius r about z_0 so that f is analytic on and inside of Γ except at z_0. Then the Laurent series for f is uniformly convergent on Γ. Hence

$$\oint_C f(z)dz = \sum_{-\infty}^{\infty} \oint_\Gamma a_n (z-z_0)^n dz$$

for $n = 0, 1, 2, \ldots$ and $n = -2, -3, \ldots$, we know that

$$\oint_\Gamma a_n (z-z_0)^n dz = 0$$

while for $n = -1$ we have

$$\oint_{\Gamma} \frac{a_{-1}}{z - z_0} dz = 2\pi i a_{-1}.$$

The Residue Theorem

Let f be single-valued and analytic inside and on a simple closed curve C except at the singularities $z_1, \ldots, z_n$ inside C having residues $a_{-1}^1, a_{-1}^2, \ldots, a_{-1}^n$. Then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} a_{-1}^k.$$

Proof: Just apply Corollary 4 of Cauchy's Theorem in conjunction with the argument given above. ∎

The calculation of Residues

1) $a_{-1} = \frac{1}{2\pi i} \oint_{\Gamma(z_0,\epsilon)} f(z)dz$, $\Gamma(z_0, \epsilon) = \{z_0 + \epsilon e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ where $\Gamma(z_0, \epsilon)$ contains no other singularities of f.
2) Let z_0 be a pole of f of order k, i.e.,

$$\lim_{z \to z_0} |(z - z_0)^k f(z)| \text{ is finite.}$$

Then

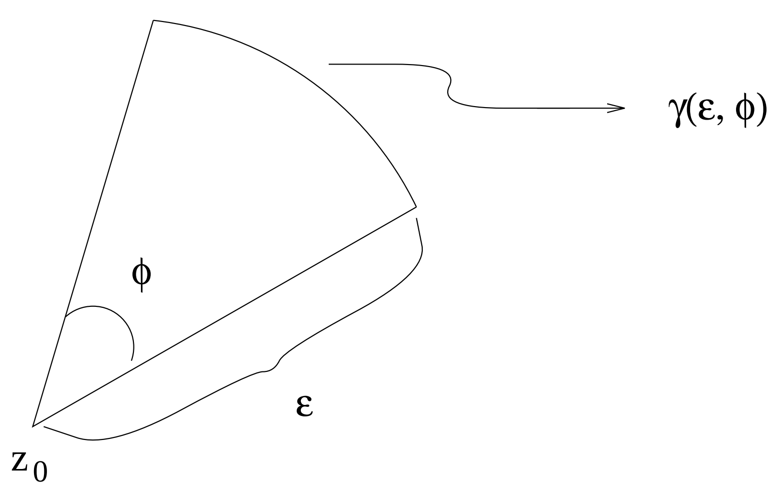
$$a_{-1} = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0)^k f(z)]$$

Thus, at a simple pole

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z).$$

Fractional Residues

Let f have a simple pole at z_0 with residue a_{-1} and let $\gamma(\epsilon, \varphi)$ be an arc of the circle $|z - z_0| = \epsilon$ that subtends an angle φ at the center.



Then

$$\lim_{\epsilon \searrow 0} \int_{\gamma} f(z)dz = i\varphi a_{-1}$$