Theory of Ordinary Differential Equations

Let $D$ be an open connected subset of $\mathbb{R} \times \mathbb{R}$ and let $f$ be a real-valued continuous function on $D$.

**ODE Problem:** Find a differentiable function $\varphi$ defined on a real interval $I$ such that

(i) $(t, \varphi(t)) \in D \quad \forall \; t \in I$, and

(ii) $\varphi'(t) = f(t, \varphi(t)) \quad \forall \; t \in I$.

The ODE Problem is called an ordinary differential equation of first order. If such an interval $I$ and function $\varphi$ exist, then it is called a solution of the differential equation

(ODE) \hspace{1cm} \dot{x} = f(t, x).

The equation (ODE) can have many solutions:

$$f(t, x) = 1$$

$\Rightarrow x = t + c$ solves (ODE) for all $c \in \mathbb{R}$. In order to fix a particular solution, certain data about the solution is specified. This gives rise to so-called “initial value” problems.

**IV Problem:** Find an interval $I$ containing $\tau$ and a solution $\varphi$ of (ODE) on $I$ satisfying

$$\varphi(\tau) = \xi.$$

This problem is denoted by

(IVP) \hspace{1cm} \dot{x} = f(t, x), x(\tau) = \xi.

Note that if $\varphi$ solves (IVP) on $I$ (with $\tau \in I$), then $\varphi$ satisfies the integral equation

$$\varphi(t) = \xi + \int_{\tau}^{t} f(s, \varphi(s)) \, ds \quad (t \in I).$$

Conversely, if $\varphi$ satisfies the integral equation, then it must be a solution to (IVP). This observation is the key to constructing solutions to (IVP).

Before proceeding we consider two examples to illustrate the type of behavior that can occur.
Example 1: \( \dot{x} = x^2, \, x(1) = -1. \)
A solution is \( \varphi(t) = -t^{-1}. \)

The corresponding interval \( I \) is \((0, \infty).\) The origin is excluded even though the function \( f(t, x) = x^2 \) is perfectly well defined there.

Thus, any existence theorem for IVP will have a local character to it. That is, if a solution exists, we can only expect it to exist near \( \tau.\)

Example 2: \( \dot{x} = x^{1/2}, \, x(0) = 0. \)
Solutions: Choose \( c \in [0, 1] \) and set

\[
\varphi_c(t) = \begin{cases} 
0 & t \in [0, c] \\
2\left(t - c\right)^{\frac{3}{2}} & t \in [c, 1]
\end{cases}.
\]

Thus, an infinite number of solutions exist on the interval \([0, 1].\)

Consequently, even if initial values are specified we still may not have a unique solution.

The existence proof proceeds in two stages.

1) Construct approximate solutions.
2) Show there exist a sequence of approximate solutions converging to a solution.

Approximate solutions:
We say that \( \varphi \) is an \( \epsilon \)-approximate solution of (ODE) on the interval \( I \) if

(i) \( (t, \varphi(t)) \in D \quad \forall \, t \in I \)
(ii) \( \varphi \in C' \) on \( I \) except for at most a finite set of points \( S \subseteq I \) where \( \varphi' \) may have simple discontinuities.
(iii) \( |\varphi'(t) - f(t, \varphi(t))| \leq \epsilon \quad \forall \, t \in I \setminus S. \)

A function satisfying (ii) is said to be piecewise continuously differentiable on \( I.\) The class of such functions is denoted \( C'_p(I). \)

Let \( (\tau, \xi) \in D.\) We construct an \( \epsilon \)-approximate solution in a neighborhood of \( \tau \) that satisfies \( \varphi(\tau) = \xi.\) We begin by specifying an interval \( I.\) Let \( R := \{(t, x): |t - \tau| \leq a, |x - \xi| \leq b\} \)
be a neighborhood of \((\tau, \xi)\) with \( R \subset D \) (\( b > 0, a > 0\)). Define

\[
M = \max |f(t, x)| \quad ((t, x) \in R)
\]

and

\[
\alpha = \min \left( a, \frac{b}{M} \right).
\]
The approximate solution that we construct is defined inside of $R$ and depends on the parameters $M$ and $\alpha$.

We construct an $\varepsilon$-approximate solution on the interval $[\tau, \tau + \alpha]$. A similar construction holds on the interval $[\tau - \alpha, \tau]$. The approximation consists of a polygonal path emanating from $(\tau, \xi)$ and possessing a finite number of straight line segments joined end to end.

Since $f$ is continuous on $R$, it is uniformly continuous on $R$. Hence there is a $\delta_\varepsilon > 0$ such that $|f(t, x) - f(s, y)| < \varepsilon$ whenever $|t - s| < \delta_\varepsilon$, $|x - y| < \delta_\varepsilon$, and $(t, x), (s, y) \in R$.

Now partition $[\tau, \tau + \alpha]$ into $n$ parts

$$\tau = t_0 < t_1 < \ldots < t_n = \tau + \alpha$$

so that

$$\max |t_k - t_{k-1}| \leq \min \left( \frac{\delta_\varepsilon}{M} \right).$$

From $(\tau, \xi)$ construct a straight line segment with slope $f(\tau, \xi)$ proceeding to the right of $\tau$ until it intersects the line $t = t_1$ at the point $(t_1, \xi_1)$. This line segment must lie in the triangle $T$. At the point $(t_1, \xi_1)$ construct a line segment with slope $f(t_1, \xi_1)$ proceeding to the right of $t_2$ until the line $t = t_2$ is intersected at the point $(t_2, \xi_2)$. Again, $(t_2, \xi_2)$ lies in the triangle $T$. Proceed in this way until the line $t = t_n = \tau + \alpha$ is intersected at $(t_n, \xi_n)$.
The resultant polygonal path yields the graph of the function $\varphi_e$;

$$\varphi_e(\tau) = \xi$$

$$\varphi_e(t) = \varphi_e(t_{k-1}) + f(t_{k-1}, \varphi_e(t_{k-1}))(t - t_{k-1})$$

$$t_{k-1} \leq t \leq t_k, l = 1, \ldots, n.$$  

Clearly, $\varphi_e \in C'_\rho[\tau, \tau + \alpha]$ and

$$|\varphi_e(t) - \varphi_e(s)| \leq M|t - s| \quad \forall \ t, s \in [\tau, \tau + \alpha].$$

Just as clearly, if $t \in [t_{k-1}, t_k]$, then

$$|\varphi'_e(t) - f(t, \varphi_e(t))| = |f(t_{k-1}, \varphi_e(t_{k-1})) - f(t, \varphi_e(t))| \leq \epsilon$$

since $|t - t_{k-1}| \leq \delta_e$. We have just established the following fact.

**Fact 1**: Given $\epsilon > 0$, there exists an $\epsilon$-approximate solution $\varphi$ of (IVP) on $|t - \tau| \leq \alpha$ such that $\varphi_e(\tau) = \xi$.

It is important to note that $\alpha$ does not depend on $\epsilon$. The idea is to let $\epsilon$ go to zero and obtain a solution to (IVP) in the limit. However, some care is required here to ensure that a limit exist.

**Definition**: A collection of real-valued functions $\mathcal{F} = \{f\}$ defined on a real interval $I$ is said to be equicontinuous on $I$ if, given $\epsilon > 0$ there exists $\delta_e > 0$, independent of $f = \mathcal{F}$, such that

$$|f(s) - f(t)| < \epsilon \text{ whenever } |s - t| < \delta_e$$

and $s, s \in I$.

Note that the family of $\epsilon$-approximate solutions to (IVP) described in Fact 1, $\{\varphi_e\}$, all satisfy

$$|\varphi_e(t) - \varphi_e(s)| \leq M|t - s| \quad \forall \ t, s \in [\tau - \alpha, \tau + \alpha].$$

Hence the $\{\varphi_e\}$ is an equicontinuous family: $\delta_e = \epsilon/M$. This observation is key in the light of the following result due to Ascoli.
Lemma. If $\mathcal{F} = \{f\}$ is a family of uniformly bounded equicontinuous functions on a bounded interval $I$, then $\mathcal{F}$ contains a sequence $\{f_n\}$, $n = 1, 2, \ldots$ that converges uniformly on $I$.

Combining Ascoli’s lemma with the observations given above, we immediately obtain the following existence theorem.

**Theorem.** (Cauchy-Peano Existence Theorem). If $f$ is continuous on a domain $D$ containing the point $(\tau, \xi)$, then there exists an interval $I$ about $\tau$ and a continuously differentiable function $\varphi$ defined on $I$ such that $\varphi$ solves (IVP);

$$\varphi'(t) = f(t, \varphi(t)) \quad \forall \ t \in I$$

$$\varphi(\tau) = \xi.$$ 

**Proof.** By Ascoli’s lemma, there is a sequence $\epsilon_n \downarrow 0$ and a function $\varphi$ defined on $[\tau-\alpha, \tau+\alpha]$ such that $\varphi_{\epsilon_n} = \varphi_n \to \varphi$ uniformly. We need to show that $\varphi$ satisfies the conclusions of the theorem. First note that

$$\varphi_n(x) = \xi + \int_\tau^t (f(s, \varphi_n(s)) + \Delta_n(s))ds$$

where $\Delta_n(t) = \varphi_n'(t) - f(t, \varphi_n(t))$ at those points where $\varphi_n'$ exists and $\Delta_n(t) = 0$ otherwise. By construction $|\Delta_n(t)| \leq \epsilon_n$. Since $f$ is uniformly continuous on $R$ and $\varphi_n \to \varphi$ uniformly on $[\tau-\alpha, \tau+\alpha]$, it follows that

$$f(t, \varphi_n(t)) \to f(t, \varphi(t))$$

uniformly on $[\tau-\alpha, \tau+\alpha]$ as $n \to \infty$. Letting $n \to \infty$, we obtain

$$\varphi(t) = \xi + \int_\tau^t f(s, \varphi(s))ds,$$

which establishes the result. \hfill \blacksquare

We now have an existence result for both (ODE) and (IVP). However, as illustrated by Example 2, the solution to (IVP) may not be unique. In order to obtain the uniqueness of a solution to (IVP), we need to impose further restrictions on the nature of the function $f$.

**Definition:** The real valued function $h$ defined on the domain $G$ is said to be Lipschitz continuous on $G$ if there exists a constant $K > 0$ such that for every pair of points $u$ and $v$ in $G$ one has

$$|h(u) - h(v)| \leq K|u - v|.$$
We say that the real-valued function \( f \) given in the statement of (ODE) satisfies a Lipschitz condition with respect to \( x \) on \( D \), if \( f \) is uniformly Lipschitz in \( x \) on \( D \), i.e., there exists \( K > 0 \) such that
\[
|f(t, x) - f(t, y)| \leq K|x - y|
\]
for all \((t, x), (t, y)\) in \( D \). We write, \( f \in (C, \text{Lip}) \) on \( D \), if \( f \in C \) on \( D \) and \( f \) is uniformly Lipschitz in \( x \) on \( D \). This Lipschitz hypothesis on \( f \) allows us to establish an important inequality concerning the \( \epsilon \)-approximate solutions of (ODE).

**Lemma A.** Suppose \( f \in (C, \text{Lip}) \) on \( D \), with Lipschitz constant \( K > 0 \). Let \( \varphi_1, \varphi_2 \) be \( \epsilon_1 \)- and \( \epsilon_2 \)-approximate solutions of (ODE) of class \( C_p' \) on some interval \((a, b)\) satisfying
\[
|\varphi_1(\tau) - \varphi_2(\tau)| \leq \delta
\]
for some \( \tau \in (a, b) \) where \( \delta > 0 \). If \( \epsilon = \epsilon_1 + \epsilon_2 \), then for all \( t \in (a, b) \),
\[
(A) \quad |\varphi_1(t) - \varphi_2(t)| \leq \delta e^{K|t-\tau|} + \frac{\epsilon}{K} \left(e^{K|t-\tau|} - 1\right).
\]

We make a few observations before establishing this lemma. First, suppose that \( \varphi_1 \) and \( \varphi_2 \) are two solutions of (IVP) so that we can take \( \delta = \epsilon_1 = \epsilon_2 = 0 \). Then inequality (A) says that
\[
\varphi_1 = \varphi_2.
\]
Consequently, (IVP) has a unique solution.

**Theorem.** (Uniqueness Theorem for (IVP)). Let \( f \in (C, \text{Lip}) \) on \( D \). If \( \varphi_1 \) and \( \varphi_2 \) are two solutions of (IVP) on the interval \( a < \tau < b \), then \( \varphi_1(t) = \varphi_2(t) \) on \((a, b)\).

Inequality (A) in Lemma (A) gives bounds on the rate at which two solutions of (ODE) can move away from each other if they were initially close to each other. That is, the bound (A) is a measure of the sensitivity of the solutions to (ODE) subject to changes in the initial conditions. This will be made more precise later. For the moment though, we address the question of how sharp this bound is.

**Example 3:** Let \( K > 0, \epsilon_1, \epsilon_2 > 0 \) and let \( P_1 = (0, \xi_1), P_2 = (0, \xi_2) \) be points in \( \mathbb{R} \times \mathbb{R} \) with \( \xi_1 \geq \xi_2 \). Let \( \varphi_1 \) and \( \varphi_2 \) be solutions of the initial value problems
\[
\dot{x} = Kx + \epsilon_1, x(0) = \xi, \text{ and } \dot{x} = Kx - \epsilon_2, x(0) = \xi_2
\]

respectively on $[0, 1]$. Then each $\varphi_i$ is an $\epsilon_i$-approximate solution for $\dot{x} = Kx$ for $i = 1, 2$, where
\[
\varphi_1(t) = \left( \xi_1 + \frac{\epsilon_1}{K} \right) e^{Kt} - \frac{\epsilon_1}{K}, \quad \text{and}
\]
\[
\varphi_2(t) = \left( \xi_2 - \frac{\epsilon_2}{K} \right) e^{Kt} + \frac{\epsilon_2}{K}.
\]
Hence
\[
|\varphi_1(t) - \varphi_2(t)| = (\xi_1 - \xi_2) e^{Kt} + \frac{\epsilon}{K} \left( e^{Kt} - 1 \right).
\]
Thus, equality can occur in inequality (A) implying that this inequality is the best that can be expected.

**Proof of Lemma A:**

We consider the case where $\tau \leq t < b$; a corresponding proof holds for $a < t \leq \tau$. We have
\[
|\varphi'_i(s) - f(s, \varphi_i(s))| \leq \epsilon_i \quad i = 1, 2
\]
extcept at possible finitely many points on $\tau \leq t < b$. Integrating this bound yields the bound
\[
|\varphi_i(t) - \varphi_i(\tau) - \int_\tau^t f(s, \varphi_i(s))ds| \leq \epsilon_i(t - \tau) \quad i = 1, 2.
\]
Combining these inequalities via the relation $|a - b| \leq |a| + |b|$, yields
\[
|\varphi_1(t) - \varphi_1(\tau) - \int_\tau^t f(s, \varphi_1(s))ds| - |\varphi_2(t) - \varphi_2(\tau) - \int_\tau^t f(s, \varphi_2(s))ds| \\
\leq \epsilon(t - \tau) \quad [\epsilon = \epsilon_1 + \epsilon_2].
\]
Define the function $r(t)$ on $[\tau, b]$ by
\[
 r(t) = |\varphi_1(t) - \varphi_2(t)|.
\]
Then, via the relation $|a| - (|b| + |c|) \leq |a - (b + c)|$, the preceding inequality yields the bound
\[
(*) \quad r(t) \leq r(\tau) + \int_\tau^t |f(s, \varphi_1(s)) - f(s, \varphi_2(s))|ds + \epsilon(t - \tau) \\
\leq r(\tau) + \int_\tau^t K r(s)ds + \epsilon(t - \tau) \\
= r(\tau) + K R(t) + \epsilon(t - \tau),
\]
where $R(t) = \int_\tau^t r(s)ds$ ($\tau \leq t < b$).
Then,

\[ R'(t) - KR(t) \leq r(\tau) + \epsilon(t - \tau) \leq \delta + \epsilon(t - \tau). \]

We now multiply both sides of this inequality by \( e^{-K(t-\tau)} \) and integrate in the variable \( t \) from \( \tau \) to \( s \in (\tau,b) \):

\[
\begin{align*}
\delta \int_{\tau}^{s} e^{-K(t-\tau)} dt + \epsilon \int_{\tau}^{s} (t - \tau)e^{-K(t-\tau)} dt \\
= \frac{\delta}{K} \left[ 1 - e^{-K(s-\tau)} \right] + \frac{\epsilon}{K^2} \int_{0}^{s} e^{-K(s-\tau)} u^2 du \\
= \frac{\delta}{K} \left[ 1 - e^{-K(s-\tau)} \right] + \frac{\epsilon}{K^2} \left[ 1 + K(s - \tau) \right] e^{-K(s-\tau)},
\end{align*}
\]

\[
\begin{align*}
\int_{\tau}^{s} e^{-K(t-\tau)} R'(t) dt - K \int_{\tau}^{s} e^{-K(t-\tau)} R(t) dt
\quad \left[ \text{Integration} \right]
\int_{\tau}^{s} e^{-K(t-\tau)} R'(t) dt - K \int_{\tau}^{s} e^{-K(t-\tau)} R(t) dt
\quad \left[ \text{by Parts} \right]
\end{align*}
\]

The inequality becomes (replacing \( s \) with \( t \))

\[ e^{-K(t-\tau)} R(t) \leq \frac{\delta}{K} \left[ 1 - e^{-K(t-\tau)} \right] + \frac{\epsilon}{K^2} \left[ 1 + K(t - \tau) \right] e^{-K(t-\tau)}. \]

or

\[ R(t) \leq \frac{\delta}{K} \left[ e^{K(t-\tau)} - 1 \right] + \frac{\epsilon}{K^2} e^{K(t-\tau)} - \frac{\epsilon}{K^2} \left[ 1 + K(t - \tau) \right]. \]

Combining this with (*) yields the result. \( \blacksquare \)

**Approximation of Solutions**

The method for approximating solutions to (IVP) with piecewise linear approximate solutions is known as Euler’s method.

We briefly consider another method of approximating solutions to (IVP). It is a straightforward application of the notion of a **fixed point iteration**. Consider the equation

\[ F(x) = x \]

where \( F : X \to X \) with \( X \) a vector space. Any solution \( \bar{x} \) to this equation is said to be a fixed point of the function \( F \). A fixed point iteration associated with this equation is given by

(FI) \[ x_{n+1} = F(x_n), \quad x_0 \text{ given.} \]
Recall that in the context of linear equations, we considered various types of fixed point iterations based on “splitting” the underlying linear operator. In the context of (IVP), consider the mapping $F : C'(I) \to C'(I)$ given by

$$F(\psi)(t) = \xi + \int_{\tau}^{t} f(s, \psi(s))ds.$$  

Note that $\varphi$ solves (IVP) if and only if

$$F(\varphi) = \varphi,$$

where $I = [\tau - \alpha, \tau + \alpha]$ with $\alpha$ as defined previously:

$$\alpha = \max \left\{ a, \frac{b}{M} \right\},$$

$$M = \max_{(t,x) \in R} |f(t, x)|,$$

and

$$R = \{(t, x) : |t - \tau| \leq a, |x - \xi| \leq b\} \subset D.$$  

Taking $\varphi_0(t) = \xi$, and applying (FI) we obtain the method of successive approximation.

**Theorem.** (Picard-Lindelöf). If $f \in (C, \text{Lip})$ on $R$, then the successive approximations $\varphi_n$ exist on $|t - \tau| \leq \alpha$ as continuous functions, and converge uniformly on this interval to the unique solution $\varphi$ of (IVP) such that $\varphi(\tau) = \xi$.

**Proof.** Again, we consider only the interval $[\tau, \tau + \alpha]$ as a similar argument holds on $[\tau - \alpha, \tau]$. It is shown by induction that each $\varphi_n$ exists, is of class $C^1$ on $[\tau, \tau + \alpha)$, and satisfies

$$|\varphi_n(t) - \xi| \leq M(t - \tau), \quad t \in [\tau, \tau + \alpha].$$

Obviously, this is all true for $\varphi_0$. Assume that it is true for each $\varphi_k$, $k = 0, 1, 2, \ldots, n$. Thus, in particular, $f(t, \varphi_n(t))$ is defined and continuous on $[\tau, \tau + \alpha]$. Then, by definition, $\varphi_{n+1}$ is of class $C'$ on $[\tau, \tau + \alpha]$ and

$$|\varphi_{n+1}(t) - \xi| \leq \int_{\tau}^{t} |f(s, \varphi_n(s))|ds \leq \int_{\tau}^{t} Mds$$

$$= M(t - \tau),$$

where the second inequality follows since

$$|\varphi_n(s) - \xi| \leq M(s - \tau) \leq M\alpha \leq b.$$
We now prove the convergence of the $\varphi_n$’s. Define

$$\Delta_n(t) = |\varphi_{n+1}(t) - \varphi_n(t)| \quad t \in [\tau, \tau + \alpha].$$

Note that

$$\Delta_n(t) \leq \int_{\tau}^{t} |f(s, \varphi_n(s)) - f(s, \varphi_{n-1}(s))|ds \leq K \int_{\tau}^{t} \Delta_{n-1}(s)ds$$

where $K$ is the uniform Lipschitz constant for $f(t, \cdot)$ over $[\tau, \tau + \alpha)$. Also,

$$\Delta_0(t) = |\varphi_1(t) - \varphi_0(t)| \leq M(t - \tau).$$

By induction, we find

$$\Delta_1(t) \leq K \int_{\tau}^{t} \Delta_0(s)ds \leq KM \frac{(t - \tau)^2}{2}$$

$$\Delta_2(t) \leq K \int_{\tau}^{t} \Delta_1(s)ds \leq MK^2 \frac{(t - \tau)^3}{6}$$

$$\vdots$$

$$\Delta_n(t) \leq MK^n \frac{(t - \tau)^{n+1}}{(n+1)!} \quad t \in [\tau, \tau + \alpha].$$
Thus, the terms of the series $\sum_{k=0}^{\infty} \Delta_n(t)$ are majorized by those for the power series for $(M/K)e^{K\alpha}$, and therefore the series $\sum_{n=0}^{\infty} \Delta_n(t)$ is uniformly convergent on $[\tau, \tau + \alpha]$. Consequently, the series

$$\varphi_0(t) + \sum_{n=0}^{\infty} (\varphi_{n+1}(t) - \varphi_n(t))$$

is absolutely and uniformly convergent on $[\tau, \tau + \alpha]$. Thus, the partial sums

$$\varphi_0(t) + \sum_{k=0}^{n-1} (\varphi_{k+1}(t) - \varphi_k(t)) = \varphi_n(t)$$

tend uniformly on $[\tau, \tau + \alpha]$ to a continuous limit function $\varphi$.

Note that $\varphi(\tau) = \xi$ since $\varphi_n(\tau) = \xi$ for all $n$. Moreover, the graph of $\varphi$ on $[\tau, \tau + \alpha]$ lies in the triangle $T$ since the graph of each $\varphi_n$ does. Thus, $f(s, \varphi(s))$ ($s \in [\tau, \tau + \alpha]$) is well defined. Finally,

$$|\int_{\tau}^{t} [f(s, \varphi(s)) - f(s, \varphi_n(s))] ds| \leq K \int_{\tau}^{t} |\varphi(s) - \varphi_n(s)| ds \to 0,$$

equally uniformly on $[\tau, \tau + \alpha]$. Hence

$$|\varphi(t) - [\xi + \int_{\tau}^{t} f(s, \varphi(s)) ds]| \leq |\varphi(t) - \varphi_{n+1}(t)| + |\int_{\tau}^{t} [f(s, \varphi(s)) - f(s, \varphi_n(s))] ds|$$

$$\to 0 \quad \text{uniformly on } [\tau, \tau + \alpha].$$

Therefore,

$$\varphi(t) = \xi + \int_{\tau}^{t} f(s, \varphi(s)) ds, \quad t \in [\tau, \tau + \alpha].$$

Since the solution is unique, this completes the proof.

It should be noted that the proof also provides a measure of the error in the approximation of $\varphi$ by $\varphi_n$:

$$|\varphi(t) - \varphi_n(t)| \leq \sum_{k=n}^{\infty} |\varphi_{k+1}(t) - \varphi_k(t)|$$

$$\leq \frac{M}{K} \sum_{k=n+1}^{\infty} \frac{K^k(t - \tau)^k}{k!}$$

$$\leq \frac{M}{K} \sum_{k=n+1}^{\infty} \frac{(K\alpha)^k}{k!}$$

$$< \frac{M}{K} \frac{(K\alpha)^{n+1}}{(n+1)!} \sum_{k=0}^{\infty} \frac{(K\alpha)^k}{k!}$$

$$= \frac{M}{K} \frac{(K\alpha)^{n+1}}{(n+1)!} e^{K\alpha}.$$
Continuation of Solutions

Suppose \( f \) is continuous on the domain \( D \subset \mathbb{R}^2 \) and that (ODE) has a solution that exists on a finite interval \((a, b)\) passing through the point \((\tau, \xi) \in D\) with \( a < \tau < b \).

We further suppose that \( f \) is bounded on \( D \), say \( M > \sup\{|f(t, x)| : (t, x) \in D\} \). Then both of the limits

\[
\varphi(a_+) = \lim_{t\downarrow 0} \varphi(a + t) \quad \text{and} \quad \varphi(b_-) = \lim_{t\uparrow 0} \varphi(b - t)
\]

exist. Indeed, since

\[
\varphi(t) = \xi + \int_{\tau}^{t} f(s, \varphi(s))ds \quad t \in (a, b),
\]

\[
|\varphi(s) - \varphi(t)| \leq \int_{t}^{s} |f(u, \varphi(u))|du \leq M|s - t|.
\]

Hence the sequences \( \{\varphi(a + t_n)\} \) and \( \{\varphi(b - t_n)\} \) are Cauchy for every sequence \( t_n \downarrow 0 \).

If the points \((a, \varphi(a_+))\) and \((b, \varphi(b_-))\) are in \( D \), we can extend the solution to the interval \([a, b]\). At the endpoints we have

\[
\varphi'_+(a) = f(a, \varphi(a_+)) \quad \text{and} \quad \varphi'_-(b) = f(b, \varphi(b_-)).
\]

Observe that we can now pose the IVP with \((\tau, \xi) = (b, \varphi(b_-))\). A solution \( \psi \) exists on some interval

\[
[b, b + \beta], \quad \beta > 0.
\]

Define

\[
\tilde{\varphi} = \varphi \quad \text{on} \quad [a, b], \quad \text{and}
\]

\[
\tilde{\varphi} = \psi \quad \text{on} \quad (b, b + \beta).
\]

Then \( \tilde{\varphi} \) is of class \( C' \) on \((a, b + \beta]\) and \( \tilde{\varphi}(\tau) = \xi \). To see this, we need only check the existence and continuity of \( \tilde{\varphi}' \) at \( b \). We have

\[
\tilde{\varphi}(t) = \varphi(b_-) + \int_{b}^{t} f(s, \tilde{\varphi}(s))ds \quad \text{on} \quad [b, b + \beta]
\]

and

\[
\varphi(b_-) = \xi + \int_{\tau}^{b} f(s, \tilde{\varphi}(s))ds.
\]

Hence

\[
\tilde{\varphi}(t) = \xi + \int_{\tau}^{t} f(s, \tilde{\varphi}(s))ds \quad \text{on} \quad (a, b + \beta].
\]

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Therefore, \( \tilde{\varphi} \) exists and is continuous at \( b \).

\( \tilde{\varphi} \) is called a continuation of \( \varphi \) to \( (a, b + \beta] \). This process can be repeated by posing the IVP

\[
\dot{x} = f(t, x), \quad x(b + \beta) = \tilde{\varphi}(b + \beta).
\]

**Theorem.** (continuation). Suppose \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) is continuous and bounded on the domain \( D \). If \( \varphi \) is a solution of (ODE) on an interval \( (a, b) \), then the limits \( \varphi(a_+) \) and \( \varphi(b_-) \) exist. If \( (a, \varphi(a_+)) \) [or \( (b, \varphi(b_-)) \)] is in \( D \), then the solution \( \varphi \) may be continued to the left of a [or the right of \( b \)].

**Systems of Differential Equations**

Suppose \( f : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n \), where \( D \) is an open connected subset of \( \mathbb{R}^{n+1} \), and consider the system of differential equations

\[
(\text{ODE}) \quad \dot{x} = f(t, x).
\]

A corresponding initial value problem is given by

\[
(\text{IVP}) \quad \dot{x} = f(t, x), \quad x(t_0) = x_0 \in \mathbb{R}^n.
\]

It turns out that all of the results obtained so far for the case \( n = 1 \) carry over to systems of equations.

We do not take the time to establish all of these extensions as they are straightforward. However, we do provide a rigorous extension of the key lemma, Lemma A. For this purpose, we need to extend the notion of an \( \varepsilon \)-approximate solution to (ODE).

We say that \( \varphi : \mathbb{R} \to \mathbb{R}^n \) is an \( \varepsilon \)-approximate solution of (ODE) on \( I \subset \mathbb{R} \) if

(i) \( (t, \varphi(t)) \in D \quad \forall t \in I \)

(ii) \( \varphi \in C^1 \) on \( I \) except for a finite set of points \( S \subset I \), and

(iii) \( \|\varphi'(t) - f(t, \varphi(t))\| \leq \varepsilon \quad \forall t \in I \setminus S \).

As in the case \( n = 1 \), we say that \( f \in (C, \text{Lip}) \) on \( D \), if \( f \) is continuous on \( D \) and there exists a constant \( K > 0 \) such that

\[
\|f(t, x) - f(t, y)\| \leq K\|x - y\|
\]

for all \( (t, x), (t, y) \in D \).
**Lemma A'.** Suppose \( f \in (C, \text{Lip}) \) on \( D \) with Lipschitz constant \( K \). Let \( \varphi_1 \) and \( \varphi_2 \) be \( \epsilon_1 \)- and \( \epsilon_2 \)-approximate solutions of (ODE) of class \( C^1_p \) on some interval \((a, b)\) such that

\[
\| \varphi_1(\tau) - \varphi_2(\tau) \| \leq \delta
\]

for some \( \tau \in (a, b) \). If \( \epsilon = \epsilon_1 + \epsilon_2 \), then for all \( t \in (a, b) \)

\[
\| \varphi_1(t) - \varphi_2(t) \| \leq \delta e^{K|t-\tau|} + \frac{\epsilon}{K} \left( e^{K|t-\tau|} - 1 \right).
\]

**Proof.** Again we only consider \( t \in [\tau, b] \). We have

\[
\| \varphi_i'(s) - f(s, \varphi_i(s)) \| \leq \epsilon_i \quad i = 1, 2 \text{ on } (a, b).
\]

Integrating from \( \tau \) to \( t \) yields

\[
\| \varphi_i(t) - \varphi_i(\tau) - \int_\tau^t f(s, \varphi_i(s))ds \| \leq \epsilon_i(t - \tau) \quad i = 1, 2.
\]

Hence,

\[
\| (\varphi_1(t) - \varphi_2(t)) - (\varphi_1(\tau) - \varphi_2(\tau)) \| - \int_\tau^t [f(s, \varphi_1(s)) - f(s, \varphi_2(s))]ds \|
\leq \epsilon(t - \tau).
\]

Setting \( r(t) = \| \varphi_i(t) - \varphi_2(t) \| \), we obtain

\[
(*) \quad r(t) \leq r(\tau) + \int_\tau^t \| f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \| ds + \epsilon(t - \tau)
\leq \delta + \int_\tau^t K r(s) ds + \epsilon(t - \tau).
\]

Setting \( R(t) = \int_\tau^t r(s)ds \), we have

\[
R'(t) - KR(t) \leq \delta + \epsilon(t - \tau).
\]

Multiplying through by \( e^{-K(t-\tau)} \) and integrating yields

\[
e^{-K(t-\tau)} R(t) \leq \frac{\delta}{K} \left( e^{-k(t-\tau)} \right) + \frac{\epsilon}{K^2} + \frac{\epsilon}{K^2} e^{K(t-\tau)} (1 + K(t - \tau))
\]

or

\[
R(t) \leq \frac{\delta}{K} \left( e^{K(t-\tau)} - 1 \right) + \frac{\epsilon}{K^2} e^{K(t-\tau)} - \frac{\epsilon}{K^2} (1 + K(t - \tau)).
\]

Combining this with \( (*) \) yields the result. \( \blacksquare \)

A particular consequence of Lemma A' is the uniqueness of the solutions to (IVP) under the assumption that \( f \in (C, \text{Lip}) \). But it is also interesting to note that Lemma A' yields an existence result for (IVP).
**Theorem.** (Existence and Uniqueness under \( f \in (C, \text{Lip}) \)). Suppose \( f \in (C, \text{Lip}) \) on the “rectangle”
\[
R := \{ (t, x) : |t - \tau| \leq a, \|x - \xi\| \leq b \} \quad (a, b > 0)
\]
and let \( M = \max \{ \| f(t, x) \| : (t, x) \in R \} \) and \( a = \min(a, b/M) \). Then there is a unique solution \( \varphi \) of class \( C' \) on \( [\tau - \alpha, \tau + \alpha] = I \) for which \( \varphi(\tau) = \xi \).

**Proof.** Let \( \epsilon_n \downarrow 0 \) as \( n \to \infty \) and let \( \varphi_n \) be an \( \epsilon_n \)-approximate solution to IVP on \( I \). Then
\[
\varphi_n(t) = \xi + \int_{\tau}^{t} (f(s, \varphi_n(s)) + \Delta_n(s)) ds
\]
where
\[
\Delta_n(t) = \varphi'_n(t) - f(t, \varphi(t))
\]
with
\[
\|\Delta_n(t)\| \leq \epsilon_n \text{ on } I \quad n = 1, 2, \ldots.
\]
Hence,
\[
\Delta_n(t) \to 0 \quad \text{uniformly on } I.
\]
Moreover, by Lemma \( A' \),
\[
\|\varphi_n(t) - \varphi_m(t)\| \leq (\epsilon_n + \epsilon_m) \left( \frac{\epsilon^K \alpha - 1}{K} \right) \quad \forall \ n, m
\]
uniformly on \( I \). Therefore, there exists a continuous function \( \varphi \) on \( I \) such that \( \varphi_n \to \varphi \) uniformly on \( I \). The continuity of \( f \) implies that
\[
f(s, \varphi_n(s)) \to f(s, \varphi(s)) \quad \text{uniformly on } I.
\]
Therefore,
\[
\varphi(t) = \xi + \int_{\tau}^{t} f(s, \varphi(s)) ds.
\]
Uniqueness follows easily from Lemma \( A' \). \[\blacksquare\]

We now focus on systems of the form
\[
(L) \quad \dot{x} = A(t)x
\]
where \( A : \mathbb{R} \to \mathbb{R}^{n \times n} \) is continuous on some (possibly infinite) interval \( I \). Because of the structure of such systems, the continuation theorem can be applied to extend any solution to the entire interval \( I \).
Theorem. (Existence and Uniqueness for (L)). For the linear system (L), where $A : [a, b] \to \mathbb{R}^{n \times n}$ is continuous, there exists a unique solution $\varphi$ on all of $[a, b]$ passing through any point $(\tau, \epsilon) \in [a, b] \times \mathbb{R}^n$ ($\varphi(\tau) = \xi$).

Proof. Clearly, $f(t, x) := A(t)x \in (C, \text{Lip})$. Hence there is a unique $\psi$ passing through $(\tau, \xi)$ and solving (L) on some subinterval $[c, d] \subseteq [a, b]$. If $c = a$ and $b = d$, we are done. Otherwise, we suppose that $I$ is a maximal subinterval of $[a, b]$ on which a continuation of $\psi$ exists. Such a maximal continuation exists by the Hausdorff Maximality Principle (such a heavy hand is not really needed here). Call this maximal continuation of $\psi$, $\hat{\psi}$. Denote the left and right endpoints of $I$ by $\alpha$ and $\beta$, respectively. We now apply Lemma A' to obtain bounds on the growth of $\hat{\psi}$. To do this we need a Lipschitz constant for $f(t, x)$ in $x$ that is uniform in $t$ on $[a, b]$. The constant $K = \max\{\|A(t)\| : t \in [a, b]\}$ will do. Taking $\varphi_1 = \hat{\psi}$ and $\varphi_2 = 0$, Lemma A' yields the bound

$$\|\hat{\psi}(t)\| \leq \|\xi\|e^{K|t-\tau|} \leq \|\xi\|e^{K(b-a)}.$$ 

Now consider the set

$$D := [a, b] \times [1 + \|\xi\|e^{K(b-a)}].$$

The function $f$ is bounded by $K[1 + \|\xi\|e^{K(b-a)}]$ on this set and graph $\tilde{\varphi} \subset D$. Hence, by our continuation theorem, $\hat{\varphi}$ can be extended outside of $(\alpha, \beta)$. This contradicts the choice of $\alpha$ and $\beta$ unless $\alpha = a$ and $\beta = b$. The uniqueness of the extension also follows from Lemma A'.

Corollary. Let $A : I \to \mathbb{R}^{n \times n}$ be continuous where $I \subset \mathbb{R}$ (possibly $I = \mathbb{R}$). Then there exist on $I$ one and only one solution $\varphi$ of (L) satisfying $\varphi(\tau) = \xi$ for $\tau \in I$ and $\xi \in \mathbb{R}^n$.

Proof. By the theorem we have existence and uniqueness on every closed subinterval of $I$ containing $\tau$. Just expand these subintervals to fill out $I$.

A matrix form of the ODE (L) yields the following result.

Theorem. (Determinant relation for matrix form of (L)). Let $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ be continuous on the interval $I = [a, b]$, and suppose $\Phi : I \to \mathbb{R}^{n \times n}$ is a matrix of functions on $I$ satisfying

$$\Phi'(t) = A(t)\Phi(t) \quad t \in I.$$

Then $\det \Phi$ satisfies on $I$ the first-order equation

$$(\det \Phi)' = (\text{tr} A)(\det \Phi)$$
and, thus for $\tau, t \in I$,

$$det \Phi(t) = det \Phi(\tau) \exp \left[ \int_{\tau}^{t} tr(A(s))ds \right].$$

**Proof.** From homework set 2, we have

$$(det \Phi)' = tr[\text{adj}(\Phi)\Phi'] = tr[\text{adj}(\Phi)A\Phi]$$

$$= tr[A\Phi\text{adj}(\Phi)] = tr[A \text{det}(\Phi)I]$$

$$= tr(A) \text{det}(\Phi),$$

where $\text{adj}(\Phi)$ denotes the classical adjoint of $\Phi$. But this is just a first order linear ODE of the form $\dot{x} = \alpha x$. The unique solution of this system is known to be

$$x = x(\tau) \exp \left[ \int_{\tau}^{t} \alpha(s)ds \right].$$

\[\blacksquare\]

The $n^{th}$-order Equation:

Let $f : \mathbb{R} \times \mathbb{R}^{s} \times \mathbb{R}^{s} \times \cdots \times \mathbb{R}^{s} \rightarrow \mathbb{R}$ be defined and continuous in a domain $D$. Then the $n^{th}$-order equation associated with $f$ is

$$(\text{ODE})_n \quad x^{(n)} = f(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}).$$

**Problem:** Find a function $\varphi$ defined on a real interval $I$ possessing $n$ derivatives there such that

(i) $$(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \in D \quad (t \in I), \text{ and}$$

(ii) $$\varphi^{(n)}(t) = f(t, \varphi(t), \ldots, \varphi^{(n-1)}(t)) \quad (t \in I).$$

If such an interval $I$ and function $\varphi$ exist, then $\varphi$ is said to be a solution to $(\text{ODE})_n$ on $I$.

The initial value problem associated with $(\text{ODE})_n$ consists of finding a solution to $(\text{ODE})_n$ passing through a specified point in $D$:

$$\varphi(\tau) = \xi_1, \varphi'(\tau) = \xi_2, \ldots, \varphi^{(n-1)}(\tau) = \xi_{n-1}.$$

The theory of $n^{th}$-order ODE’s can be reduced to that of first-order ODE’s in the following way:
Define $K \in \mathbb{R}^{ns \times ns}$ by

$$K = \begin{bmatrix}
O_{s \times s} & I_{s \times s} & O_{s \times s} & \cdots & \cdots & O_{s \times s} \\
O_{s \times s} & O_{s \times s} & I_{s \times s} & \cdots & \vdots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
O_{s \times s} & \cdots & \cdots & I_{s \times s} & O_{s \times s}
\end{bmatrix}$$

and $F : \mathbb{R} \times \mathbb{R}^{ns} \to \mathbb{R}^{ns}$ by

$$F(t, x) = Kx + \hat{f}(t, x)$$

where $\hat{f}(t, x) = [0, 0, \ldots, 0, f(t, x)]^T$, and consider the system $\dot{x} = F(t, x)$. If $\varphi = [\varphi_1, \varphi_2, \ldots, \varphi_n]^T$ with $\varphi_j : I \to \mathbb{R}^s$, $j = 1, \ldots, n$ solves this system, then

$$\varphi''_{j-1} = \varphi_j, \quad j = 2, \ldots, n,$$

and

$$\varphi_1^{(n)} = f(t, \varphi_1, \varphi''_1, \varphi_{1}^{(n-1)}).$$

Therefore, $\varphi_1$ solves (ODE)$_n$. Conversely, if $\varphi_1$ solves (ODE)$_n$, then $\varphi = [\varphi_1, \varphi'_1, \ldots, \varphi_1^{(n-1)}]$ solves

$$\dot{x} = F(t, x).$$

Dependence of Solutions on Initial Conditions

A solution of an (ODE) on an interval $I$ can be considered as a function not only of $t \in I$, but also of the coordinates of a point through which it passes.

**Example:** The ODE $\dot{x} = x$ has solutions

$$\varphi(t) = \xi e^{t-\tau}$$

where $\varphi(\tau) = \xi$.

This observation determines a function of $(t, \tau, \xi)$ which we also call $\varphi$: e.g.

$$\varphi(t, \tau, \xi) = \xi e^{t-\tau}.$$ 

We wish to investigate the properties of $\varphi$. Is it continuous, differentiable, twice differentiable, Lipschitz continuous? We begin with the following result.

**Theorem.** (Solvability for Perturbed IVP’s). Let $f \in (C, \text{Lip})$ in a domain $D \subset \mathbb{R}^{n+1}$, and suppose $\psi$ is a solution of (ODE) on an interval $I : a \leq t \leq b$. Then there exists $\delta > 0$ such that for any $(\tau, \xi) \in U$,

$$U = \{(\tau, \xi) : a < \tau < b, ||\xi - \psi(\tau)|| < \delta\}$$

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there exists a unique solution $\varphi$ of (ODE) on $I$ with $\varphi(\tau, \tau, \xi) = \xi$. Moreover, $\varphi$ is continuous on the set

$$V := \{(t, \tau, \xi) : a < t < b, (\tau, \xi) \in U\}.$$

**Remark:** An important special case of this result is when $\tau$ is fixed and

$$U = \{\xi : \|\xi - \psi(\tau)\| < \delta\}$$

$$V = \{(t, \xi) : a < t < b, \xi \in U\}.$$

A consequence of the theorem in this case is that to each $t \in (a, b)$ we obtain a mapping $T_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by

$$T_t(\tau, \xi) = (t, \varphi(t, \tau, \xi)).$$

For each $t \in (a, b)$, the mapping $T_t$ is a homeomorphism, i.e., one-to-one with both $T_t$ and $T_t^{-1}$ continuous. The fact that $T_t$ is invertible is a consequence of the uniqueness of solutions. The continuity of $T_t$ follows from the continuity of $\varphi$ in $\tau$ and $\xi$. Finally, since

$$\xi = \varphi(\tau, t, \varphi(t, \tau, \xi)),$$

we have

$$T_t^{-1}(t, \xi) = (\tau, \varphi(\tau, t, \xi)) = T_{\tau}(t, \xi)$$

so that the continuity of $T_t^{-1}$ also follows from that of $\varphi$.

**Proof of Theorem:**

Choose $\delta_1 > 0$ so that the “tube”

$$U_1 = \{(t, x) : t \in I, \|x - \psi(t)\| \leq \delta_1\}$$

is in $D$. 
Next let $\delta > 0$ be chosen so that

$$e^{K|b-a|}\delta < \delta_1$$

(this is the $\delta$ to be used in the application of Lemma $A'$). Let

$$U = \{(\tau, \xi) : a < \tau < b, \|\xi - \psi(\tau)\| < \delta\}.$$ 

If $(\tau, \xi) \in U$, there exists a solution $\varphi$ through $(\tau, \xi)$ satisfying

$$\varphi(t, \tau, \xi) = \xi + \int_\tau^t f(s, \varphi(s, \tau, \xi))ds$$

for $\tau \in [c, d]$ with $a \leq c < d \leq b$. By Lemma $A'$, we have

$$\|\varphi(t, \tau, \xi) - \psi(t)\| \leq \|\xi - \psi(\tau)\|e^{K|d-c|} < \delta_1$$

so that graph $\varphi(\cdot, \tau, \xi) \subset U_1$. Therefore, by our continuation result, $\varphi$ can be extended to all of $I$. The continuity of $\varphi$ is now established by showing that $\varphi$ is the uniform limit of continuous functions.

Set

$$\varphi_0(t, \tau, \xi) = \psi(t) + \xi - \psi(\tau)$$

$$\varphi_{j+1}(t, \tau, \xi) = \xi + \int_\tau^t f(s, \varphi_j(s, \tau, \xi))ds \quad j = 0, 1, 2, \ldots$$
This is a sequence of successive approximations based on a fixed point iteration.

By induction, we show that

(i) \( \varphi_j \) is continuous on

\[ V = \{(t, \tau, \xi) : t \in I, (\tau, \xi) \in U \}, \]

(ii) for all \( (\tau, \xi) \in U, \varphi_j(t, \tau, \xi) \in U \) for \( t \in I \), and

(iii) \( \|\varphi_j(t, \tau, \xi) - \varphi_j(t, \tau)\| \leq \frac{K^j|t-\tau|}{(j+1)!}\|\xi - \psi(\tau)\| \) for all \( (t, \tau, \xi) \in V \).

We begin with the \( j = 0 \) case. Clearly, (i) holds and (ii) follows from the inequality

\[ \|\varphi_0(t, \tau, \xi) - \psi(t)\| = \|\xi - \psi(\tau)\| < \delta_1. \]

To see (iii), note that

\[
\|\varphi_1(t, \tau, \xi) - \varphi_0(t, \tau, \xi)\| = \left\| \int_\tau^t (f(s, \varphi_0(s, \tau, \xi)) - f(s, \psi(s))) ds \right\|
\leq K \int_\tau^t |\varphi_0(s, \tau, \xi) - \psi(s)| ds = K|t-\tau|\|\xi - \psi(\tau)\|.
\]

Now assume that (i)–(iii) hold for \( j = 0, 1, \ldots, n \). Then (i) holds at \( j = n + 1 \) by the definition of \( \varphi_{n+1} \). By (iii) for \( j = 0, 1, \ldots, n \), we have

\[
\|\varphi_{n+1}(t, \tau, \xi) - \psi(t)\| \leq \sum_{j=0}^{n} \|\varphi_{j+1}(t, \tau, \xi) - \psi_j(t, \tau, \xi)\|
+ \|\varphi_0(t, \tau, \xi) - \psi(t)\|
\leq \sum_{j=0}^{n+1} \frac{K^j|t-\tau|}{j!}\|\xi - \psi(\tau)\| \leq e^{K|t-\tau|}\delta < \delta_1.
\]

This establishes (ii) for \( j = n + 1 \). Finally,

\[
\|\varphi_{n+2}(t, \tau, \xi) - \varphi_{n+1}(t, \tau, \xi)\| = \left\| \int_\tau^t f(s, \varphi_{n+1}(s, \tau, \xi)) - f(s, \varphi_n(s, \tau, \xi)) ds \right\|
\leq K \int_\tau^t |\varphi_{n+1}(s, \tau, \xi) - \varphi_n(s, \tau, \xi)| |ds|
\leq \frac{K^{n+2}|t-\tau|}{(n+2)!}\|\xi - \psi(\tau)\|
\]

establishing (iii) for \( j = n + 1 \).

A telescoping series argument using (iii) above shows that the sequence \( \varphi_j \) converges uniformly on \( V \) to \( \varphi \). This establishes the continuity of \( \varphi \) on \( V \). \[ \square \]
Having established the continuity of the mapping \( \varphi(t, \tau, \xi) \), we now consider its differentiability. Particular attention is paid to differentiability in the \( \xi \) argument.

Before proceeding to the main result, we review some facts from finite dimensional analysis.

**Definition.** (Modulus of Continuity). Given a continuous function \( F : D \to \mathbb{R}^m \) where \( D \subset \mathbb{R}^n \), we define the modulus of continuity of \( F \) on \( D \) by

\[
\omega(t) := \sup\{\|F(x) - F(y)\| : \|x - y\| \leq t, \ x, \ y \in D\}
\]

**Fact:** \( F \) is uniformly continuous on \( D \) if and only if \( \omega(t) \to 0 \) as \( t \to 0 \).

**Lemma.** let \( D \) be a compact convex subset of \( \mathbb{R}^n \) with nonempty interior. If \( F : D \to \mathbb{R}^m \) is continuously differentiable on \( D \), then

\[
\|F(y) - [F(x) + F'(x)(y - x)]\| \leq \omega(\|x - y\|)\|x - y\|
\]

where \( \omega \) is the modulus of continuity for \( F' \) on \( D \).

**Proof.** Given \( x, y \in D \) we have

\[
\|F(y) - [F(x) + F'(x)(y - x)]\| = \| \int_0^1 [F'(x + t(y - x)) - F'(x)](y - x) dt \|
\]

\[
\leq \int_0^1 \|F'(x + t(y - x)) - F'(x)\|\|y - x\| dt \leq \omega(\|y - x\|)\|y - x\|.
\]

**Theorem.** (Differentiability of \( \varphi(t, \tau, \xi) \)). Let the hypotheses of the previous Theorem stand. In addition suppose that \( f_x \) exists and is continuous on \( D \). Then \( \varphi \) is continuously differentiable on the set

\[
V = \{(t, \tau, \xi) : a < t < b, a < \tau < b, \|\xi - \psi(\tau)\| < \delta\}
\]

where \( \psi \) and \( \delta \) are designated by the statement of the previous result. Moreover,

\[
\det \varphi_\xi(t, \tau, \xi) = \exp \left[ \int_\tau^t \text{tr}[f_x(s, \varphi(s, \tau, \xi))] ds \right].
\]

**Remark:** The assumption that \( f \) is continuously differentiable on \( D \) makes the hypothesis \( f \in (C, \text{Lip}) \) redundant.
Proof. To establish that $\varphi \in C'$ on $V$, it is sufficient to show that each of the partials
\( \partial \varphi / \partial t, \partial \varphi / \partial \tau, \partial \varphi / \partial \xi_j \) exists and is continuous on $V$. This is how we shall proceed.

First observe that the existence and continuity of $\partial \varphi / \partial t$ follows trivially from the definition of $\varphi$. Thus, we need only consider the partials $\frac{\partial}{\partial z_j} \varphi$ where $z = (\tau, \xi)$.

The motivation for the proof is very simple. Since $\varphi$ solves (ODE), we have

$$\varphi'(t, z) = f(t, \varphi(t, z)).$$

Differentiating with respect to $z_j$ yields

$$\left[ \frac{\partial}{\partial z_j} \varphi \right]'(t, z) = f_x(t, \varphi(t, z)) \frac{\partial}{\partial z_j} \varphi(t, z).$$

Therefore, the function $u_j(t) = \frac{\partial}{\partial z_j} \varphi(t, z)$, if it exists, satisfies the linear ODE

\[(*)^1 \quad u_j'(t) = A(t)u_j(t),\]

where $A(t) = f_x(t, \varphi(t, z))$. By our results for linear systems, a solution to $(*)^1$ exist on $[a, b]$ for any consistent set of initial conditions. We need to do two things in the argument to follow:

1. specify the initial conditions for $(*)^1$ that yield $u_j$ as a solution.
2. Show that $u_j(t, z)$ is continuous.

Let us first consider the $\xi$ arguments. Choose $h \in \mathbb{R}^n$ and set

$$\theta(t, \tau, \xi, \alpha) := \frac{\varphi(t, \tau, \xi + \alpha h) - \varphi(t, \tau, \xi)}{\alpha}.$$

By Lemma $A'$, we have

$$\| \theta(t, \tau, \xi, \alpha) \| \leq \| \theta(t, \tau, \mu, \xi, \alpha) \| e^{K|b-a|} = \| h \| e^{K|b-a|}.$$

Moreover, $\theta$ satisfies

$$\theta'(t, \tau, \xi, \alpha) = \frac{f(t, \varphi(t, \tau, \xi + \alpha h)) - f(t, \varphi(t, \tau, \xi))}{\alpha}.$$

Thus, if $\omega$ is the modulus of continuity for $f$ on $V$, we have, by the preceding lemma,

$$\| \theta'(t, \tau, \xi, \alpha) - f_x(t, \varphi(t, \tau, \xi)) \theta(t, \tau, \xi, \alpha) \|
\leq \omega(\| h \| e^{K|b-a|}) \| h \| e^{K|b-a|} \leq \| h \| \omega(\| \xi \| \| h \| e^{K|b-a|}) e^{K|b-a|}.$$
Therefore, \( \theta(t, \tau, \xi, \alpha) \) is a \( \mu(\alpha) \)-approximate solution of the (IVP)

\[
\begin{cases}
y' = f_x(t, \varphi(t, \tau, \xi))y \\
y(\tau) = \theta(\tau, \tau, \xi, 0) = h.
\end{cases}
\]

where \( \mu(\alpha) = \|h\|\omega(|\alpha|\|h\|e^{b-a})e^{K(b-a)}. \)

Let \( \beta(t, \tau, \xi) \) be the solution of \((*)^2\) on \([a, b]\). Then, by Lemma A,

\[
(*)^3 \quad \|\theta(t, \tau, \xi, \alpha) - \beta(t, \tau, \xi)\| \leq \frac{\mu(\alpha)}{K}(e^{K(b-a)} - 1)
\]

for all \((t, \tau, \xi) \in V\) and \( \alpha \) sufficiently small. Moreover, by the previous theorem, \( \beta \) is continuous on \( V \). The inequality \((*)^3\) shows that the limit

\[
\lim_{\alpha \to 0} \theta(t, \tau, \xi, \alpha) =: \theta(t, \tau, \xi, 0)
\]

exists with \( \theta(t, \tau, \xi, \alpha) \) converging to \( \beta(t, \tau, \xi) \) uniformly on \( V \). Therefore, \( \theta(t, \tau, \xi, 0) \) is continuous on \( V \). Setting \( h = e_j \) (the \( j \)th canonical basis vector), we have

\[
\theta(t, \tau, \xi, 0) = \frac{\partial}{\partial \xi_j} \varphi(t, \tau, \xi).
\]

It now only remains to establish the existence and continuity of \( \frac{\partial}{\partial \tau} \varphi \) on \( V \). First note that

\[
\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi) = \varphi(t, \tau + \alpha, \xi) - \xi
\]

\[
= \varphi(t, \tau + \alpha, \xi) - \varphi(\tau + \alpha, \tau + \alpha, \xi)
\]

\[
= \int_{\tau + \alpha}^T f(s, \varphi(s, \tau + \alpha, \xi)) ds.
\]

Therefore,

\[
(*)^4 \quad \|\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi)\| \leq M|\alpha|,
\]

and

\[
(*)^5 \quad \frac{\partial}{\partial \tau} \varphi(t, \tau, \xi) = \lim_{\alpha \to 0} \frac{\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi)}{\alpha} = -f(t, \xi).
\]

From Lemma A’ and \((*)^4\), we find that

\[
\|\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi)\| \leq \|\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi)\|e^{K(b-a)}
\]

\[
(*)^6 \quad \leq M|\alpha|e^{K(b-a)}.
\]

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Setting \( \chi(t, \tau, \xi, \alpha) = \frac{\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi)}{\alpha} \), we have

\[
\chi'(t, \tau, \xi, \alpha) = \frac{f(t, \varphi(t, \tau + \alpha, \xi)) - f(t, \varphi(t, \tau, \xi))}{\alpha}.
\]

Therefore, by the preceeding lemma and \((*)^4\),

\[
\|\chi'(t, \tau, \xi, \alpha) - f_x(t, \varphi(t, \tau, \xi)) \chi(t, \tau, \xi, \alpha)\|
\leq \omega(\|\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi)\|) \left( \frac{\|\varphi(t, \tau + \alpha, \xi) - \varphi(t, \tau, \xi)\|}{|\alpha|} \right)
\leq \omega(M|\alpha|)M.
\]

Continuing just as before, except now with \( \beta \) being the solution to

\[
y' = f_x(t, \varphi(t, \tau, \xi))y \\
y(\tau) = -f(\tau, \xi),
\]

we find that \( \beta \) is the uniform limit of \( \chi(t, \tau, \xi, \alpha) \) on \( V \) as \( \alpha \to 0 \) so that \( \frac{\partial}{\partial \tau} \varphi(t, \tau, \xi) \) exists and is continuous on \( V \).

The final statement in the theorem is an immediate consequence of our result on linear systems of the form

\[\Phi'(t) = A(t)\Phi(t).\]

**Parametrized ODE’s**

Let \( f : D \to \mathbb{R}^n \) be continuous where \( D \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \) is a domain and consider the parametrized family of ODE’s

(ODE)\(_u\)

\[
\dot{x} = f(t, x, u)
\]

where \( u \in D_0 = \{ u \in \mathbb{R}^k : \exists (\tau, \zeta) \text{ s. t. } (\tau, \zeta, u) \in D \}. \)

**Theorem.** Let \( u_0 \in D_0 \) and let \( \psi \) be a solution of (ODE)\(_u_0\) on \( I : a \leq t \leq b. \) Then there exists a \( \delta > 0 \) such that for any \( (\tau, \zeta, u) \in U_0 \) where

\[
U_0 = \{ (\tau, \zeta, u) : a < \tau < b, \|\zeta - \psi(\tau)\| + \|u - u_0\| < \delta \}.
\]

There exist a unique solution \( \varphi \) of (ODE)\(_u\) on \( I \) satisfying

\[
\varphi(\tau, \tau, \zeta, u) = \zeta.
\]
Moreover, \( \varphi \) is continuous on
\[
V_0 = \{(t, \tau, \zeta, u) : a < t < b, (\tau, \zeta, u) \in U_0 \}.
\]

**Proof.** Again the proof follows by successive approximation.

Choose \( \delta_1 > 0 \) so that
\[
U_1 = \{(t, x, u) : a \leq t \leq b, \|x - \psi(t)\| + \|u - u_0\| \leq \delta_1 \}
\]
is contained in \( D \).

Choose \( (t, x, u) \) from \( U_1 \) and define successive approximations by
\[
\varphi_0(t, \tau, \zeta, u) = \psi(t) + \zeta - \psi(\tau)
\]
\[
\varphi_{j+1}(t, \tau, \zeta, u) = \zeta + \int_{\tau}^{t} f(s, \varphi_j(s, \tau, \zeta, u), u) ds
\]
Clearly,
\[
\|\varphi_0(t, \tau, \zeta, u) - \psi(t)\| = \|\zeta - \psi(\tau)\|
\]
and
\[
\|\varphi_1(t, \tau, \zeta, u) - \varphi_0(t, \tau, \zeta, u)\| = \left\| \int_{\tau}^{t} [f(s, \varphi_0(s, \tau, \zeta, u), u) - f(s, \psi(s), u_0)] ds \right\|.
\]
The uniform continuity of \( f \) in \( U_1 \) implies the existence of a \( \delta_\epsilon > 0 \) for each \( \epsilon > 0 \) such that
\[
\|f(s, \varphi_0(s, \tau, \zeta, u), u) - f(s, \psi(s), u)\| < \epsilon
\]
provided
\[
a \leq s \leq b, \quad (\tau, \zeta, u) \in U_1, \text{ and}
\]
\[
\|\zeta - \psi(\tau)\| + \|u - u_0\| < \delta_\epsilon.
\]
Hence,
\[
\|\varphi_1(t, \tau, \zeta, u) - \varphi_0(t, \tau, \zeta, u)\| < \epsilon|t - \tau|
\]
if \( \|\zeta - \psi(\tau)\| + \|u - u_0\| < \delta_\epsilon \).

Proceeding by induction, we obtain
\[
\|\varphi_{j+1}(t, \tau, \zeta, u) - \varphi_j(t, \tau, \zeta, u)\| \leq \frac{\epsilon|t - \tau|^{j+1} K^j}{(j + 1)!}.
\]

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Choose $\varepsilon > 0$ so that
\[
\frac{\varepsilon}{k} (e^{K(b-a)} - 1) \leq \frac{\delta_1}{2}
\]
and let
\[
\delta = \min\{\delta_1, \delta_1/2\}.
\]
Then for all $j = 0, 1, \ldots$ graph$(\varphi_j(\tau, \zeta, u)) \subset U_1$. The uniformity of the convergence yields the result.

Differentiability results can also be extended to the setting of parametrized ODE’s.

**Theorem.** Let the hypotheses of the previous theorem stand and suppose that both $f_x$ and $f_u$ exist and are continuous on $D_0$. Then the solution mapping $\varphi$ given by the previous theorem is continuously differentiable on $V_0$.

**Proof.** Let
\[
w_i = x_i \quad i = 1, \ldots, n
\]
\[
w_{i+n} = u_i \quad i = 1, \ldots, k
\]
and
\[
F_i(t, u) = f_i(t, x, u) \quad i = 1, \ldots, n
\]
\[
F_{i+n}(t, u) = 0 \quad i = 1, \ldots, k.
\]
Consider the (ODE)
\[
\dot{w} = F(t, u).
\]
This (ODE) satisfies the conditions necessary to obtain the differentiability of the solution mapping, $\chi(t, \tau, (x, u))$. Set
\[
\chi_i(t, \tau, x, u) = \varphi_i(t, \tau, x, u) \quad i = 1, \ldots, n
\]
\[
\chi_{n+i}(t, \tau, x, u) = u_i \quad i = 1, \ldots, k
\]
with $\chi_i(\tau, \tau, x, u) = \zeta_i, i = 1, \ldots, u$. The mapping $\varphi$ so determined yields the result.

**Complex Systems**

Let $f : D \to \mathbb{C}^n$ with $D \subset \mathbb{R} \times \mathbb{C}^n$

\[
\dot{w} = f(t, w)
\]

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All existence, uniqueness, continuation, and dependence results carry over. To see this, think of \( w \) as an element of \( \mathbb{R}^{2n} \) by considering the behavior of the real and imaginary parts separately.

Next suppose \( f : D \to \mathbb{C}^n \) with \( D \subset \mathbb{C} \times \mathbb{C}^n \). We assume that \( f \) is analytic, that is,

\[
f_j(s, w)
\]

is an analytic function of each of its components \( s, w_1, \ldots, w_n \) for \( j = 1, \ldots, n \). Thus, \( f \) possesses all derivatives of all orders. Consider the ODE

\[
(\text{ODE}_c) \quad \dot{w} = f(s, w)
\]

where \( \dot{w} = \frac{\partial}{\partial s} w \).

We say that \( \varphi \) is a solution to \( (\text{ODE}_c) \) if there is a domain \( H \subset \mathbb{C} \) such that

1. \( (\xi, \varphi(\xi)) \in D \quad \forall \xi \in H \), and

2. \( \varphi'(\xi) = f(\xi, \varphi(\xi)) \quad \forall \xi \in H. \)

Although there is a fundamental difference between \( (\text{ODE}) \) and \( (\text{ODE}_c) \) one can apply the same techniques of proof. The key observation in this regard is that \( \varphi \) solves \( (\text{ODE}_c) \) if and only if

\[
(\text{IE}) \quad \varphi(t) = \varphi(\xi_0) + \int_{\xi_0}^{\xi} f(z, \varphi(z)) dz
\]

where the above integral is taken along any path in \( H \) connecting \( \xi_0 \) to \( \xi \). This equivalence and the fact that \( (\text{IE}) \) is well-defined follows immediately from the analyticity of \( f \). Therefore, the method of successive approximation can be applied just as before to yield results comparable to those obtained in the real case. A sample result of this type follows.

**Theorem.** *(Existence and Uniqueness for \( (\text{ODE}_c) \).)* Suppose \( f \) is bounded and analytic on the open rectangle

\[
R = \{ (\xi, w) : |\xi - \xi_0| < a, \| w - w_0 \| < b \}
\]

and set

\[
M = \sup_{R} \| f(\xi, w) \|, \ 0 < \alpha < \min(a, b/M).
\]
Then there exists on the disk $|\xi - \xi_0| < \alpha$ a unique analytic function $\varphi$ solving $(ODE_c)$ and satisfying $\varphi(\xi_0) = w_0$.

**Proof.** Since $f_w$ is bounded on any closed subrectangle $R_1$ of $R$, $f$ is Lipschitz on $R_1$. Let $K_1$ be the Lipschitz constant. Construct the successive approximations

$$\varphi_0(\xi) = w_0$$

$$\varphi_{j+1}(\xi) = \varphi_j(\xi_0) + \int_{\xi_0}^{\xi} f(z, \varphi_j(z))dz.$$

Clearly, each $\varphi_j$ is analytic. We show by induction that

$$\|\varphi_j(\xi) - w_0\| \leq M |\xi - \xi_0|.$$  

This is clearly true for $j = 0$. Assume it is true for $j = 1, \ldots, N$. Then

$$\|\varphi_{N+1}(\xi) - \zeta\| \leq \int_{\xi_0}^{\xi} \|f(z, \varphi_N(z))\| |dz|$$

$$\leq M |\xi - \xi_0|$$

where the second inequality follows since

$$\|\varphi_N(\xi) - \zeta\| \leq M |\xi - \xi_0| \leq M \alpha < b.$$  

We now prove the convergence of the $\varphi_j$’s. Define

$$\Delta_j(\xi) = \|\varphi_{j+1}(\xi) - \varphi_j(\xi)\|, \ |\xi - \xi_0| < \alpha.$$  

Then

$$\Delta_j(\xi) \leq \int_{\xi_0}^{\xi} \|f(x, \varphi_j(z)) - f(z, \varphi_{j-1}(z))\| |dz|$$

$$\leq K \int_{\xi_0}^{\xi} \Delta_{j-1}(\xi)|dz|.$$  

Also, $\Delta_0(\xi) = \|\varphi_1(\xi) - \varphi_0(\xi)\| \leq M |\xi - \xi_0|$. Thus, by induction,

$$\Delta_1(\xi) \leq K \int_{\xi_0}^{\xi} \Delta_0(z)|dz| \leq KM |\xi - \xi_0|^2$$

$$\Delta_2(\xi) \leq K \int_{\xi_0}^{\xi} \Delta_1(z)|dz| \leq MK^2 |\xi - \xi_0|^3$$

$$\vdots$$

$$\Delta_j(\xi) \leq MK^j \frac{|\xi - \xi_0|^{j+1}}{(j+1)!},$$  

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whenever $|\xi - \xi_0| < \alpha$. Thus, the terms of the series $\sum_{j=0}^{\infty} \Delta_j(\xi)$ are majorized by those for the power series for $\frac{M}{K} e^{K\alpha}$. Therefore, the series is uniformly convergent on $|\xi - \xi_0| < \alpha$. Consequently, the series

$$\varphi_0(\xi) + \sum_{j=0}^{\infty} \varphi_{j+1}(\xi) = \varphi_j(\xi) = \lim_{j \to \infty} \varphi_j(\xi)$$

is absolutely and uniformly convergent on $|\xi - \xi_0| < \alpha$. Denote the uniform limit by $\varphi$. Since each $\varphi_j$ is analytic so must $\varphi$ be analytic. Also, $\varphi(\xi_0) = w_0$ since $\varphi_j(\xi_0) = w_0$ for all $j = 0, 1, 2, \ldots$. Moreover, $(\xi, \varphi(\xi)) \in R$ for all $\xi$ satisfying $|\xi - \xi_0| < \alpha$ since

$$\|\varphi_j(\xi) - w_0\| \leq M\alpha < b$$

for all $j = 0, 1, 2, \ldots$. Finally,

$$\| \int_{\xi_0}^{\xi} [f(z, \varphi(z)) - f(z, \varphi_j(z))] dz \|$$

$$\leq K \int_{\xi_0}^{\xi} \| \varphi(z) - \varphi_j(z) \| |dz| \to 0$$

uniformly on $|\xi - \xi_0| < \alpha$. Hence

$$\| \varphi(\xi) - [w_0 + \int_{\xi_0}^{\xi} f(z, \varphi(z)) dz] \|$$

$$\leq \| \varphi(\xi) - \varphi_{j+1}(\xi) \| + \| \int_{\xi_0}^{\xi} [f(z, \varphi(z)) - f(z, \varphi_j(z))] dz \|$$

$$\to 0 \quad \text{uniformly on } |\xi - \xi_0| < \alpha.$$ 

Therefore, $\varphi(\xi) = w_0 + \int_{\xi_0}^{\xi} f(z, \varphi(z)) dz$ on $|\xi - \xi_0| < \infty$. The uniqueness of the solutions follows from an extension of Lemma A' to the complex case.  

\[\square\]