Resolvent

Let V be a finite-dimensional vector space and $L \in \mathcal{L}(V)$. If $\zeta \notin \sigma(L)$, then the operator $L - \zeta I$ is invertible, so we can form

$$R(\zeta) = (L - \zeta I)^{-1}$$

(which we sometimes denote by $R(\zeta, L)$). The function $R : \mathbb{C}\backslash\sigma(L) \to \mathcal{L}(V)$ is called the *resolvent* of L. It provides an analytic approach to questions about the spectral theory of L. The set $\mathbb{C}\backslash\sigma(L)$ is called the *resolvent set* of L. Since the inverses of commuting invertible linear transformations also commute, $R(\zeta_1)$ and $R(\zeta_2)$ commute for $\zeta_1, \zeta_2 \in \mathbb{C}\backslash\sigma(L)$. Since a linear transformation commutes with its inverse, it also follows that L commutes with all $R(\zeta)$.

We first want to show that $R(\zeta)$ is a holomorphic function of $\zeta \in \mathbb{C}\backslash\sigma(L)$ with values in $\mathcal{L}(V)$. Recall our earlier discussion of holomorphic functions with values in a Banach space; one of the equivalent definitions was that the function is given by a norm-convergent power series in a neighborhood of each point in the domain. Observe that

$$\begin{aligned} R(\zeta) &= (L - \zeta I)^{-1} \\ &= (L - \zeta_0 I - (\zeta - \zeta_0)I)^{-1} \\ &= (L - \zeta_0 I)^{-1}[I - (\zeta - \zeta_0)R(\zeta_0)]^{-1}. \end{aligned}$$

Let $\|\cdot\|$ be a norm on V, and $\|\cdot\|$ denote the operator norm on $\mathcal{L}(V)$ induced by this norm. If

$$|\zeta - \zeta_0| < \frac{1}{\|R(\zeta_0)\|},$$

then the second inverse above is given by a convergent Neumann series:

$$\begin{aligned} R(\zeta) &= R(\zeta_0)\sum_{k=0}^{\infty} R(\zeta_0)^k(\zeta - \zeta_0)^k \\ &= \sum_{k=0}^{\infty} R(\zeta_0)^{k+1}(\zeta - \zeta_0)^k. \end{aligned}$$

Thus $R(\zeta)$ is given by a convergent power series about any point $\zeta_0 \in \mathbb{C}\backslash\sigma(L)$ (and of course the resolvent set $\mathbb{C}\backslash\sigma(L)$ is open), so $R(\zeta)$ defines an $\mathcal{L}(V)$-valued holomorphic function on the resolvent set $\mathbb{C}\backslash\sigma(L)$ of L. Note that from the series one obtains that

$$\left(\frac{d}{dS}\right)^k R(\zeta)\bigg|_{\zeta_0} = k!R(\zeta_0)^{k+1}.$$

Hence for any $\zeta \in \mathbb{C}\backslash\sigma(L)$,

$$\left(\frac{d}{dS}\right)^k R(\zeta) = k!R(\zeta)^{k+1}.$$

This can be remembered easily by noting that it follows formally by differentiating $R(\zeta) = (L - \zeta)^{-1}$ with respect to ζ, treating L as a parameter.

The argument above showing that $R(\zeta)$ is holomorphic has the advantage that it generalizes to infinite dimensions. Although the following alternate argument only applies in finite dimensions, it gives stronger results in that case. Let $n = \dim V$, choose a basis for V, and represent L by a matrix in $\mathbb{C}^{n\times n}$, which for simplicity we will also call L. Then the matrix of $(L-\zeta I)^{-1}$ can be calculated using Cramer's rule. First observe that

$$\det(L-\zeta I) = (-1)^n p_L(\zeta).$$

Also each of the components of the classical adjoint matrix of $L-\zeta I$ is a polynomial in ζ of degree at most $n-1$. It follows that each component of $(L-\zeta I)^{-1}$ is a rational function of ζ (which vanishes at ∞), so in that sense $R(\zeta)$ is a rational $\mathcal{L}(V)$-valued function. Also each eigenvalue λ_i of L is a pole of $R(\zeta)$ of order at most m_i, the algebraic multiplicity of λ_i. Of course $R(\zeta)$ cannot have a removable singularity at $\zeta = \lambda_i$, for otherwise letting $\zeta \to \lambda_i$ in the equation $(L-\zeta I)R(\zeta) = I$ would show that $L-\lambda_i I$ is invertible, which it is not.

We calculated above the Taylor expansion of $R(\zeta)$ about any point $\zeta_0 \in \mathbb{C}\backslash\sigma(L)$. It is also useful to calculate the Laurent expansion about the poles. Recall the spectral decomposition of L: if $\lambda_1,\ldots,\lambda_k$ are the distinct eigenvalues of L with algebraic multiplicities $m_1,\ldots,m_k$, and

$$\widetilde{E}_i = \mathcal{N}((L-\lambda_i I)^{m_i})$$

are the generalized eigenspaces, then

$$V = \bigoplus_{i=1}^k \widetilde{E}_i,$$

and each $\widetilde{E}_i$ is invariant under L. Let $P_1,\ldots,P_k$ be the associated projections, so that

$$I = \sum_{i=1}^k P_i.$$

Let $N_1,\ldots,N_k$ be the associated nilpotent transformations. We may regard each N_i as an element of $\mathcal{L}(V)$ (in which case

$$N_i = P_i N P_i$$

where

$$N = N_1 + \cdots + N_k,$$

so

$$N_i[\widetilde{E}_i] \subset \widetilde{E}_i \text{ and } N_i[\widetilde{E}_j] = 0 \text{ for } j \neq i),$$

or we may regard N_i as its restriction to $\widetilde{E}_i$ with

$$N_i : \widetilde{E}_i \to \widetilde{E}_i.$$

Now

$$L = \sum_{i=1}^k \lambda_i P_i + N_j,$$

so

$$L - \zeta I = \sum_{i=1}^{k} [(\lambda_i - \zeta)P_i + N_i].$$

Clearly to invert $L - \zeta I$, it suffices to invert each $(\lambda_i - \zeta)P_i + N_i$ on $\widetilde{E}_i$. But on $\widetilde{E}_i$,

$$(\lambda_i - \zeta)P_i + N_i = (\lambda_i - \zeta)[I - (\zeta - \lambda_i)^{-1} N_i].$$

For a nilpotent operator N with $N^m = 0$,

$$(I - N)^{-1} = I + N + N^2 + \cdots + N^{m-1}.$$

This is a special case of a Neumann series which converges since it terminates. Thus

$$\left([(\lambda_i - \zeta)P_i + N_i] \Big|_{\widetilde{E}_i} \right)^{-1} = (\lambda_i - \zeta)^{-1} \sum_{\ell=0}^{m_i - 1} (\zeta - \lambda_i)^{-\ell} N_i^{\ell} = - \sum_{\ell=0}^{m_i - 1} (\zeta - \lambda_i)^{-\ell-1} N_i^{\ell}.$$

The direct sum of these operators gives $(L - \zeta I)^{-1}$, so we obtain

$$R(\zeta) = - \sum_{i=1}^{k} \left[(\zeta - \lambda_i)^{-1} P_i + \sum_{\ell=0}^{m_i - 1} (\zeta - \lambda_i)^{-\ell-1} N_i^{\ell} \right].$$

This result is called the *partial fractions decomposition* of the resolvent. Recall that any rational function $q(\zeta)/p(\zeta)$ with $\deg q < \deg p$ has a unique partial fractions decomposition of the form

$$\sum_{i=1}^{k} \left[\sum_{j=1}^{m_i} \frac{a_{ij}}{(\zeta - r_i)^j} \right]$$

where $a_{ij} \in \mathbb{C}$ and

$$p(\zeta) = \prod_{i=1}^{k} (\zeta - r_i)^{m_i}$$

is the factorization of p (normalized to be monic, r_i distinct). The above is such a decomposition for $R(\zeta)$.

Observe that the partial fractions decomposition gives the Laurent expansion of $R(\zeta)$ about all of its poles all at once: about $\zeta = \lambda_i$ the holomorphic part of $R(\zeta)$ is the sum over all other eigenvalues. In particular, for the coefficients of $(\zeta - \lambda_i)^{-1}$ and $(\zeta - \lambda_i)^{-2}$ we have

$$\mathcal{R}\underset{\zeta=\lambda_i}{es}[R(\zeta)] = -P_i \qquad \text{and} \qquad \mathcal{R}\underset{\zeta=\lambda_i}{es}[(\zeta - \lambda_i)R(\zeta)] = -N_i.$$

So the full spectral decomposition of L is encoded in $R(\zeta)$. It is in fact possible to give a complete treatment of the spectral problem — including a proof of the spectral decomposition — based purely on a study of the resolvent and its properties. Beginning with the fact that $R(\zeta)$ has poles at the λ_i's, one can show that for each i, $-\mathcal{R}\underset{\zeta=\lambda_i}{es}[R(\zeta)]$ is a projection and $-\mathcal{R}\underset{\zeta=\lambda_i}{es}[(\zeta - \lambda_i)R(\zeta)]$ is nilpotent, that the sum of the projections is the identity, etc. See Kato for such a treatment.

The special case of the partial fractions decomposition in which L is diagonalizable is particularly easy to derive and remember. If L is diagonalizable then each $\widetilde{E}_i = E_{\lambda_i}$ is the eigenspace and each $N_i = 0$. If $v \in V$, we may write

$$x = \sum_{i=1}^{k} v_i \text{ (uniquely) where } v_i = P_i v \in E_{\lambda_i}.$$

Then $Lv = \sum_{i=1}^{k} \lambda_i v_i$, so

$$(L - \zeta I)v = \sum_{i=1}^{k} (\lambda_i - \zeta) v_i,$$

so clearly

$$R(\zeta)v = \sum_{i=1}^{k} (\lambda_i - \zeta)^{-1} p_i v,$$

and thus

$$R(\zeta) = \sum_{i=1}^{k} (\lambda_i - \zeta)^{-1} P_i.$$

The powers $(\lambda_i - \zeta)^{-1}$ arise from inverting $(\lambda_i - \zeta)I$ on each E_{λ_i}.

We discuss briefly two applications of the resolvent — each of these has many ramifications which we do not have time to investigate fully. Both applications involve contour integration of operator-valued functions. If $M(\zeta)$ is a continuous function of ζ with values in $\mathcal{L}(V)$ and γ is a C^1 contour in $\mathbb{C}$, we may form $\int_\gamma M(\zeta)d\zeta \in \mathcal{L}(V)$. This can be defined by choosing a fixed basis for V, representing $M(\zeta)$ as matrices, and integrating componentwise, or as a norm-convergent limit of Riemann sums of the parameterized integrals. By considering the componentwise definition it is clear that the usual results in complex analysis automatically extend to the operator-valued case, for example if $M(\zeta)$ is holomorphic in a neighborhood of the closure of a region bounded by a closed curve γ except for poles $\zeta_1, \ldots, \zeta_k$, then $\frac{1}{2\pi i}\int_\gamma M(\zeta)d\zeta = \sum_{i=1}^{k} \mathcal{R}es(M, \zeta_i)$.

Perturbation of Eigenvalues and Eigenvectors

One major application of resolvents is the study of perturbation theory of eigenvalues and eigenvectors. We sketch how resolvents can be used to study continuity properties of eigenvectors. Suppose $A_t \in \mathbb{C}^{n\times n}$ is a family of matrices depending continuously on a parameter t. (In our examples, the domain of t will be a subset of $\mathbb{R}$, but in general the domain of t could be any metric space.) It is a fact that the eigenvalues of A_t depend continuously on t, but this statement must be properly formulated since the eigenvalues are only determined up to order. Since the eigenvalues are the roots of the characteristic polynomial of A_t, and the coefficients of the characteristic polynomial depend continuously on t, (since, by norm equivalence, the entries of A_t depend continuously on t), it suffices to see that the roots of a monic polynomial (of fixed degree) depend continuously on is coefficients. Consider first the case of a simple root: suppose $z_0 \in \mathbb{C}$ is a simple root of a polynomial p_0. We may choose a closed disk about z_0 containing no other zero of p_0; on the boundary γ of this disk p_0 does not vanish, so all polynomials with coefficients sufficiently close to those of p_0 also do not vanish on γ. So for such p, $p'(z)/p(z)$ is continuous on γ, and by the argument principle,

$$\frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}dz$$

is the number of zeroes of p (including multiplicities) in the disk. For p_0, we get 1. Since $p \neq 0$ on γ, $\frac{p'}{p}$ varies continuously with the coefficients of p, so

$$\frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}dz$$

also varies continuously with the coefficients of P. As it is integer-valued we conclude that it must be the constant 1, so all nearby polynomials have exactly one zero in the disk. Now the residue theorem gives that

$$\frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}z\,dz = z_p$$

is the unique root of p in the disk. As the left hand side varies continuously with p, it follows that its simple root z_p does too.

One can also obtain information near multiple zeroes using such arguments. If z_0 is a root of p_0 of multiplicity $m > 1$, then it follows as above that in *any* sufficiently small disk about z_0, any polynomial p sufficiently close to p_0 (where "sufficiently close" depends on the radius of the disk) will have exactly m zeroes in that disk (counting multiplicities). This is one sense in which it can be said that the eigenvalues depend continuously on the coefficients. There are stronger senses as well.

However, eigenvectors do not generally depend continuously on parameters. Consider for example the family given by

$$A_t = \begin{bmatrix} t & 0 \\ 0 & -1 \end{bmatrix} \text{ for } t \geq 0 \text{ and } A_t = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} \text{ for } t \leq 0.$$

For each t, the eigenvalues of A_t are t, $-t$. Clearly A_t is diagonalizable for all t. But it is impossible to find a continuous function $v : \mathbb{R} \to \mathbb{R}^2$ such that $v(t)$ is an eigenvector of A_t for each t. For $t > 0$, the eigenvectors of A_t are multiples of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

while for $t < 0$ they are multiples of

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix};$$

clearly it is impossible to join up such multiples continuously by a vector $v(t)$ which doesn't vanish at $t = 0$. (Note that a similar C^∞ example can be constructed: let

$$A_t = \begin{bmatrix} \varphi(t) & 0 \\ 0 & -\varphi(t) \end{bmatrix} \text{ for } t \geq 0, \text{ and } A_t = \begin{bmatrix} 0 & \varphi(t) \\ \varphi(t) & 0 \end{bmatrix} \text{ for } t \leq 0,$$

where

$$\varphi(t) = e^{-1/|t|} \text{ for } t \neq 0 \text{ and } \varphi(0) = 0 \text{ .)}$$

In the example above, A_0 has an eigenvalue of multiplicity 2. We show, using the resolvent, that if an eigenvalue of A_0 has algebraic multiplicity 1, then the corresponding eigenvector can be chosen to depend continuously on t, at least in a neighborhood of $t = 0$. Suppose λ_0 is an eigenvalue of A_0 of multiplicity 1. We know from the above that A_t has a unique eigenvalue λ_t near λ_0 for t near 0; moreover λ_t is simple and depends continuously on t for t near 0. If γ is a circle about λ_0 as above, and we set

$$R_t(\zeta) = (A_t - \zeta I)^{-1},$$

then

$$-\frac{1}{2\pi i}\int_\gamma R_t(\zeta)d\zeta = -\mathcal{R}es_{\zeta=\lambda_t} R_t(\zeta) = P_t,$$

where P_t is the spectral projection onto the 1-dimensional eigenspace of A_t corresponding to λ_t. Observe that for t near 0 and $\zeta \in \gamma$, $A_t - \zeta I$ is invertible, and it is clear that $R_t(\zeta)$ depends continuously on t (actually, uniformly in $\zeta \in \gamma$). So P_t depends continuously on t for t near 0. We can obtain a continuously-varying eigenvector by projecting a fixed vector: let v_0 be a unit eigenvector for A_0 corresponding to λ_0, and set

$$v_t = P_t v_0 = -\frac{1}{2\pi i}\int_\gamma R_t(\zeta) v_0 d\zeta \ .$$

The right hand side varies continuously with t, so v_t does too and

$$v_t\Big|_{t=0} = v_0.$$

Hence $v_t \neq 0$ for t near 0, and since v_t is in the range of $P)t$, v_t is an eigenvector of A_t corresponding to λ_t, as desired.

Remark. These ideas can show that if A_t is a C^k function of t, i.e. each $a_{ij}(t)$ has k continuous derivatives, then also λ_t and v_t are C^k functions of t.

This approach using the resolvent indicates that it is possible to obtain something continuous even when there are multiple eigenvalues. As long as no eigenvalues of A_t hit γ, the expression $-\frac{1}{2\pi i}\int_\gamma R_t(\zeta)d\zeta$ depends continuously on t. By the Residue Theorem, for each t this is the sum of the projections onto all generalized eigenspaces corresponding to eigenvalues in the disk enclosed by γ, so this sum of projections is always continuous.

Spectral Radius

We now show how the resolvent can be used to give a formula for the spectral radius of an operator which does not require knowing the spectrum explicitly; this is sometimes useful. As before, let $L \in \mathcal{L}(V)$ where V is finite dimensional. Then

$$R(\zeta) = (L - \zeta I)^{-1}$$

is a rational $\mathcal{L}(V)$ - valued function of ζ with poles at the eigenvalues of L. In fact, from Cramers rule we saw that

$$R(\zeta) = \frac{Q(\zeta)}{p_L(\zeta)},$$

where $Q(\zeta)$ is an $\mathcal{L}(V)$ - valued polynomial in ζ of degree $\leq n-1$ and p_L is the characteristic polynomial of L. Since $\deg p_L > \deg Q$, it follows that $R(\zeta)$ is holomorphic to ∞ and vanishes there; i.e., for large $|\zeta|$, $R(\zeta)$ is given by a convergent power series in $\frac{1}{\zeta}$ with zero constant term. We can identify the coefficients in this series (which are $\in \mathcal{L}(V)$) using Neumann series: for $|\zeta|$ sufficiently large,

$$R(\zeta) = -\zeta^{-1}(I - \zeta^{-1}L)^{-1} = -\zeta^{-1}\sum_{k=0}^{\infty}\zeta^{-k}L^k = -\sum_{k=0}^{\infty}L^k\zeta^{-k-1} .$$

The coefficients in the expansion are (minus) the powers of L. For any submultiplicative norm on $\mathcal{L}(V)$, this series converges for $\|\zeta^{-1}L\| < 1$, i.e., for $|\zeta| > \|L\|$.

Recall from complex analysis that the radius of convergence r of a power series

$$\sum_{k=0}^{\infty} a_k z^k$$

can be characterized in two ways: first, as the radius of the largest open disk about the origin in which the function defined by the series has a holomorphic extension, and second directly in terms of the coefficients by the formula

$$\frac{1}{r} = \overline{\lim}_{k\to\infty}|a_k|^{\frac{1}{k}}.$$

These characterizations also carry over to operator-valued series

$$\sum_{k=0}^{\infty} A_k z^k \qquad (\text{where } A_k \in \mathcal{L}(V)).$$

Such a series also has a radius of convergence r, and both characterizations generalize: the first is unchanged; the second becomes

$$\frac{1}{r} = \overline{\lim}_{k\to\infty}\|A_k\|^{k}.$$

Note that the expression

$$\overline{\lim}_{k\to\infty}\|A_k\|^{\frac{1}{k}}$$

is independent of the norm on $\mathcal{L}(V)$ by the Norm Equivalence Theorem since $\mathcal{L}(V)$ is finite-dimensional. These characterizations in the operator-valued case can be obtained by considering the series for each component in any matrix realization.

Apply these two characterizations to the power series

$$\sum_{k=0}^{\infty} L^k\zeta^{-k} \quad \text{in} \quad \zeta^{-1}$$

for $-\zeta R(\zeta)$. We know that $R(\zeta)$ is holomorphic in $|\zeta| > \rho(L)$ (including at ∞) and that $R(\zeta)$ has poles at each eigenvalue of L, so the series converges for $|\zeta| > \rho(L)$, but in no larger disk about ∞. The second formula gives

$$\{\zeta : |\zeta| > \overline{\lim}_{k\to\infty}\|L^k\|^{\frac{1}{k}}\}$$

as the largest disk of convergence, and thus

$$\rho(L) = \overline{\lim}_{k\to\infty}\|L^k\|^{\frac{1}{k}}.$$

Lemma. If $L \in \mathcal{L}(V)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated with multiplicities, then the eigenvalues of L^ℓ are $\lambda_1^\ell, \ldots, \lambda_n^\ell$.

Remark. This is a special case of the Spectral Mapping Theorem which we will study soon.

Proof. If L has spectral decomposition

$$L = \sum_{i=1}^{k}(\mu_i P_i + N_i)^\ell = \sum_{i=1}^{k}(\mu_i^\ell P_i + N_i'),$$

where

$$N_i'' = \sum_{j=1}^{\ell} \left(\begin{array}{c} \ell \\ j \end{array} \right) \mu_i^{\ell-j} N_i^j$$

is nilpotent. The result follows from the uniqueness of the spectral decomposition. □

Remark. Alternate proofs can be based on the Schur Triangularization Theorem, or by Jordan form, using a basis of V for which the matrix of L is upper triangular. The diagonal elements of a power of a triangular matrix are that power of the diagonal elements of the matrix. The result follows.

Proposition. If $\dim V < \infty$, $L \in \mathcal{L}(V)$, and $\|\cdot\|$ is any norm on $\mathcal{L}(V)$, then

$$\rho(L) = \lim_{k\to\infty} \|L^k\|^{\frac{1}{k}}.$$

Proof. We have already shown that

$$\rho(L) = \overline{\lim}_{k\to\infty}\|L^k\|^{\frac{1}{k}},$$

so we just have to show the limit exists. By norm equivalence, the limit exists in one norm iff it exists in every norm, so it suffices to show the limit exists if $\|\cdot\|$ is submultiplicative. Let $\|\cdot\|$ be submultiplicative. Then $\rho(L) \leq \|L\|$. By the lemma,

$$\rho(L^k) = \rho(L)^k \quad \text{so} \quad \rho(L)^k = \rho(L^k) \leq \|L^k\|.$$

Thus

$$\rho(L) \leq \underline{\lim}_{k\to\infty} \|L^k\|^{\frac{1}{k}} \leq \overline{\lim}_{k\to\infty}\|L^k\|^{\frac{1}{k}} = \rho(L).$$

, so the limit exists and is $\rho(L)$. □

This formula for the spectral radius $\rho(L)$ of L allows us to extend the class of operators in $\mathcal{L}(V)$ for which we can guarantee that certain series converge. Recall that if $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$ is holomorphic for $|z| < r$ and $L \in \mathcal{L}(V)$ satisfies $\|L\| < r$ for some submultiplicative norm, then $\varphi(L)$ can be defined as the limit of the norm-convergent series $\sum_{k=0}^{\infty} a_k L^k$. In fact, this series converges under the (apparently weaker) assumption that $\rho(L) < r$: choose $\epsilon > 0$ so that $\rho(L) + \epsilon < r$; for k sufficiently large, $\|L^k\|^{\frac{1}{k}} \leq \rho(L) + \epsilon$, so

$$\sum_{k \text{large}} \|a_k L^k\| \leq \sum |a_k|(\rho(L)+\epsilon)^k < \infty.$$

For example, the Neumann series

$$(I-L)^{-1} = \sum_{k=0}^{\infty} L^k$$

converges whenever $\rho(L) < 1$. It may happen that $\rho(L) < 1$ and yet $\|L\| > 1$ for certain natural norms (like the operator norms induced by the ℓ^p norms on $\mathbb{C}^n$, $1 \leq p \leq \infty$). An extreme case occurs when L is nilpotent, so $\rho(L) = 0$, but $\|L\|$ can be large (e.g. the matrix

$$\begin{bmatrix} 0 & 10^{10} \\ 0 & 0 \end{bmatrix};$$

in this case, of course, any series $\sum_{k=0}^{\infty} a_k L^k$ converges since it terminates finitely.

The following question has arisen a couple of times in the discussion of the spectral radius: given a *fixed* $L \in \mathcal{L}(V)$, what is the infimum of $\|L\|$ as $\|\cdot\|$ ranges over all submultiplicative norms on $\mathcal{L}(V)$? What if we only consider operator norms on $\mathcal{L}(V)$ induced by norms on V? How about restricting further to operator norms on $\mathcal{L}(V)$ induced by inner products on V? We know that $\rho(L) \leq \|L\|$ in these situations. It turns out that the infimum in each of these situations is actually $\rho(L)$.

Proposition. Given $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$, there exists a norm $\|\cdot\|$ on $\mathbb{C}^n$ for which, in the operator norm induced by $\|\cdot\|$, we have $\|A\| \leq \rho(A) + \epsilon$.

Caution: The norm depends on A and ϵ.

Proof. Choose an invertible matrix $S \in \mathbb{C}^{n\times n}$ for which

$$J = S^{-1}AS$$

is in Jordan form. Write $J = \Lambda + Z$, where

$$\Lambda = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$$

is the diagonal part of J and Z is a matrix with only zero entries except possibly for some one(s) on the first superdiagonal ($i = j+1$). Let

$$D = \text{diag}\,(1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}).$$

Then

$$D^{-1}JD = \Lambda + \epsilon Z.$$

Fix any p with $1 \le p \le \infty$. Then in the operator norm $|||\cdot|||_\rho$ on $\mathbb{C}^{n\times n}$ induced by the ℓ^p-norm $\|\cdot\|_p$ on $\mathbb{C}^n$,

$$|||\Lambda|||_p = \max\{|\lambda_j| : 1 \le j \le n\} = \rho(A)$$

and $|||Z|||_p \le 1$, so

$$|||\Lambda + \epsilon Z|||_p \le \rho(A) + \epsilon.$$

Define $\|\cdot\|$ on $\mathbb{C}^n$ by

$$\|x\| = \|D^{-1}S^{-1}x\|_p.$$

Then

$$\begin{aligned} \|A\| &= \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{y\neq 0} \frac{\|ASDy\|}{\|SDy\|} = \sup_{y\neq 0} \frac{\|D^{-1}S^{-1}ASDy\|_p}{\|y\|_p} \\ &= \|\Lambda + \epsilon Z|||_p \le \rho(A) + \epsilon \ . \end{aligned}$$

□

Exercise: Show that we can choose an inner product on $\mathbb{C}^n$ which induces such a norm.

Remarks.

(1) This proposition is easily extended to $L \in \mathcal{L}(V)$ for $\dim V < \infty$.

(2) This proposition gives another proof that if $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$ is holomorphic for $|z| < r$ and $L \in \mathcal{L}(V)$ satisfies $\rho(L) < r$, then the series $\sum_{k=0}^{\infty} a_k L^k$ converges: choose $\epsilon > 0$ so that $\rho(L) + \epsilon < r$, and then choose a submultiplicative norm on $\mathcal{L}(V)$ for which $\|L\| \le \rho(L) + \epsilon$; then $\|L\| < r$ and the series converges.

(3) One can use the Schur Triangularization Theorem instead of Jordan form in the proof; see Lemma 5.6.10 in H-J.

We conclude this discussion of the spectral radius with two corollaries of the formula

$$\rho(L) = \lim_{k\to\infty} \|L^k\|^{\frac{1}{k}} \quad \text{for} \quad L \in \mathcal{L}(V)$$

with $\dim V < \infty$.

Corollary. $\rho(L) < 1$ iff $L^k \to 0$.

Proof. By norm equivalence, we may use a submultiplicative norm on $\mathcal{L}(V)$. If $\rho(L) < 1$, choose $\epsilon > 0$ with $\rho(L) + \epsilon < 1$. For large k, $\|L^k\| \le (\rho(L) + \epsilon)^k \to 0$ as $k \to \infty$. Conversely, if $L^k \to 0$, then $\exists\, k \ge 1$ with $\|L^k\| < 1$, so $\rho(L^k) < 1$, so by the lemma, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of L all satisfy $\|\lambda_j^k| < 1$ and thus $\rho(L) < 1$. □

Corollary. $\rho(L) < 1$ iff there is a submultiplicative norm on $\mathcal{L}(V)$ an and an integer $k \ge 1$ such that $\|L^k\| < 1$.

Functional Calculus

Our last application of resolvents is to define functions of an operator. We do this using a method providing good operational properties, so this is called a functional "calculus."

Let $L \in \mathcal{L}(V)$ and suppose that φ is holomorphic in a neighborhood of the closure of a bounded open set $\Delta \subset \mathbb{C}$ with C^1 boundary satisfying $\sigma(L) \subset \Delta$. For example, Δ could be a large disk containing all the eigenvalues of L, or the union of small disks about each eigenvalue, or an appropriate annulus centered at $\{0\}$ if L is invertible. Give the curve $\partial\Delta$ the orientation induced by the boundary of Δ (i.e. the winding number $n(\partial\Delta, z) = 1$ for $z \in \Delta$ and $= 0$ for $z \in \mathbb{C}\backslash\bar{\Delta}$.) We define $\varphi(L)$ by requiring that the Cauchy integral formula for φ should hold.

Definition.

$$\varphi(L) = -\frac{1}{2\pi i}\int_{\partial\Delta}\varphi(\zeta)R(\zeta)d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta}\varphi(\zeta)(\zeta I - L)^{-1}d\zeta.$$

We first observe that the definition of $\varphi(L)$ is independent of the choice of Δ. In fact, since $\varphi(\zeta)R(\zeta)$ is holomorphic except for poles at the eigenvalues of L, we have by the residue theorem that

$$\varphi(L) = -\sum_{i=1}^{k}\mathcal{R}es_{\zeta=\lambda_i}[\varphi(\zeta)R(\zeta)],$$

which is clearly independent of the choice of Δ. In the special case where $\Delta_1 \subset \Delta_2$, it follows from Cauchy's theorem that

$$\int_{\partial\Delta_2}\varphi(\zeta)R(\zeta)d\zeta - \int_{\partial\Delta_1}\varphi(\zeta)R(\zeta)d\zeta = \int_{\partial(\Delta_2\backslash\bar{\Delta}_1)}\varphi(\zeta)R(\zeta)d\zeta = 0$$

since $\varphi(\zeta)R(\zeta)$ is holomorphic in $\Delta_2\backslash\bar{\Delta}_1$. This argument can be generalized as well.

Next, we show that this definition of $\varphi(L)$ agrees with previous definitions. For example, suppose φ is the constant function 1. Then the residue theorem gives

$$\varphi(L) = -\sum_{i=1}^{k}\mathcal{R}es_{\zeta=\lambda_i}R(\zeta) = \sum_{i=1}^{k}P_i = I.$$

If $\varphi(\zeta) = \zeta^n$ for an integer $n > 0$, then take Δ to be a large disk containing $\sigma(L)$ with boundary γ, so

$$\begin{aligned}
\varphi(L) &= \frac{1}{2\pi i}\int_\gamma \zeta^n(\zeta I - L)^{-1}d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma [(\zeta I - L) + L]^n(\zeta I - L)^{-1}d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{j=0}^{n}\binom{n}{j}L^j(\zeta I - L)^{n-j-1}d\zeta \\
&= \sum_{j=0}^{n}\binom{n}{j}L^j\int_\gamma(\zeta I - L)^{n-j-1}d\zeta\ .
\end{aligned}$$

For $j < n$, the integrand is holomorphic in Δ, so by Cauchy theorem

$$\int_\gamma (\zeta I - L)^{n-j-1} d\zeta = 0.$$

For $j = n$, we obtain

$$\int_\gamma (\zeta I - L)^{-1} d\zeta = I$$

as above, so $\varphi(L) = L^n$ as desired. It follows that this new definition of $\varphi(L)$ agrees with the usual definition of $\varphi(L)$ when φ is a polynomial.

Consider next the case in which

$$\varphi(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k,$$

where the series converges for $|\zeta| < r$. We have seen that if $\rho(L) < r$, then the series $\sum_{k=0}^{\infty} a_k L^k$ converges in norm. We will show that this definition of $\varphi(L)$ (via the series) agrees with our new definition of $\varphi(L)$ (via contour integrating). Choose

$$\Delta \subset \{\zeta : |\zeta| < r\} \text{ with } \sigma(L) \subset \Delta \text{ and } \gamma \equiv \partial\Delta \subset \{\zeta : |\zeta| < r\}.$$

We want to show that

$$\frac{-1}{2\pi i} \int_\gamma \varphi(\zeta) R(\zeta) d\zeta = \sum_{k=0}^{\infty} a_k L^k.$$

Set

$$\varphi_N(\zeta) = \sum_{k=0}^{N} a_k \zeta^k.$$

Then $\varphi_N \to \varphi$ uniformly on compact subsets of $\{\zeta : |\zeta| < r\}$; in particular $\varphi_N \to \varphi$ uniformly on γ. If $A(t)$ is a continuous $\mathcal{L}(V)$-valued function of $t \in [a, b]$, then for any norm on $\mathcal{L}(V)$,

$$\left\| \int_a^b A(t) dt \right\| \leq \int_a^b \|A(t)\| dt$$

(this follows from the triangle inequality applied to the Riemann sums approximating the integrals upon taking limits). So

$$\left\| \int_\gamma (\varphi(\zeta) - \varphi_N(\zeta)) R(\zeta) d\zeta \right\| \leq \int_\gamma |\varphi(\zeta) - \varphi_N(\zeta)| \cdot \|R(\zeta)\| d|\zeta|.$$

Since $\|R(\zeta)\|$ is bounded on γ, it follows that

$$\lim_{N\to\infty} \int_\gamma \varphi_N(\zeta) R(\zeta) d\zeta = \int_\gamma \varphi(\zeta) R(\zeta) d\zeta$$

in norm. But φ_N is a polynomial, so

$$-\frac{1}{2\pi i} \int_\gamma \varphi_N(\zeta) R(\zeta) d\zeta = \varphi_N(L)$$

as above. Thus

$$-\frac{1}{2\pi i}\int_\gamma \varphi(\zeta)R(\zeta)d\zeta = \lim_{N\to\infty}\left(-\frac{1}{2\pi i}\int_\gamma \varphi_N(\zeta)R(|z)d\zeta\right) = \lim_{N\to\infty}\varphi_N(L) = \sum_{k=0}^{\infty} a_k L^k,$$

and the two definitions of $\varphi(L)$ agree.

Operational Properties

Lemma. (The First Resolvent Equation)
If $L \in \mathcal{L}(V)$, $\zeta_1, \zeta_2 \notin \sigma(L)$, and $\zeta_1 \neq \zeta_2$, then

$$R(\zeta_1)\circ R(\zeta_2) = \frac{R(\zeta_1) - R(\zeta_2)}{\zeta_1 - \zeta_2}.$$

Proof.

$$R(\zeta_1) - R(\zeta_2) = R(\zeta_1)(L - \zeta_2 I)R(\zeta_2) - R(\zeta_1)(L - \zeta_1 I)R(\zeta_2) = (\zeta_1 - \zeta_2)R(\zeta_1)R(\zeta_2).$$

□

Proposition. Suppose $L \in \mathcal{L}(V)$ and φ_1 and φ_2 are both holomorphic in a neighborhood of $\sigma(L)$. Then

(a) $(a_1\varphi_1 + a_2\varphi_2)(L) = a_1\varphi_1(L) + a_2\varphi_2(L)$, and

(b) $(\varphi_1\varphi_2)(L) = \varphi_1(L)\circ\varphi_2(L)$.

Proof. (a) follows immediately from the linearity of contour integration. By the lemma,

$$\begin{aligned}
\varphi_1(L)\circ\varphi_2(L) &= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\varphi_1(\zeta_1)R(\zeta_1)d\zeta_1 \circ \int_{\gamma_2}\varphi_2(\zeta_2)R(\zeta_2)d\zeta_2 \\
&= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}\varphi(\zeta_1)\varphi(\zeta_2)R(\zeta_1)\circ R(\zeta_2)d\zeta_2 d\zeta_1 \\
&= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}\varphi(\zeta_1)\varphi(\zeta_2)\frac{R(\zeta_1) - R(\zeta_2)}{\zeta_1 - \zeta_2}d\zeta_2 d\zeta_1.
\end{aligned}$$

Thus far, γ_1 and γ_2 could be any curves encircling $\sigma(L)$; the curves could cross and there is no problem since

$$\frac{R(\zeta_1) - R(\zeta_2)}{\zeta_1 - \zeta_2}$$

extends to $\zeta_1 = \zeta_2$. However, we want to split up the $R(\zeta_1)$ and $R(\zeta_2)$ pieces, so we need to make sure the curves don't cross. For definiteness, let γ_1 be the union of small circles around each eigenvalue of L, and let γ_2 be the union of slightly larger circles. Then

$$\varphi_1(L)\circ\varphi_2(L) = \frac{1}{(2\pi i)^2}\left[\int_{\gamma_1}\varphi_1(\zeta_1)R(\zeta_1)\int_{\gamma_2}\frac{\varphi_2(\zeta_2)}{\zeta_1 - \zeta_2}d\zeta_2 d\zeta_1 - \int_{\gamma_2}\varphi_2(\zeta_2)R(\zeta_2)\int_{\gamma_1}\frac{\varphi_1(\zeta_1)}{\zeta_1 - \zeta_2}d\zeta_1 d\zeta_2\right].$$

Since ζ_1 is inside γ_2 but ζ_2 is outside γ_1,

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{\varphi_2(\zeta_2)}{\zeta_2-\zeta_1}d\zeta_2 = v_2(\zeta_1) \quad \text{and} \quad \frac{1}{2\pi i}\int_{\gamma_1}\frac{\varphi_1(\zeta_1)}{\zeta_1-\zeta_2}d\zeta_1 = 0,$$

so

$$\varphi_1(L)\circ\varphi_2(L) = -\frac{1}{2\pi i}\int_{\gamma_1}\varphi_1(\zeta_1)\varphi_2(\zeta_1)R(\zeta_1)d\zeta_1 = (\varphi_1\varphi_2)(L),$$

as desired. □

Remark. Since $(\varphi_1\varphi_2)(\zeta) = (\varphi_2\varphi_1)(\zeta)$, (b) implies that $\varphi_1(L)$ and $\varphi_2(L)$ always commute.

Example. Suppose $L \in \mathcal{L}(V)$ is invertible and $\varphi(\zeta) = \frac{1}{\zeta}$. Since $\sigma(L) \subset \mathbb{C}\backslash\{0\}$ and φ is holomorphic on $\mathbb{C}\backslash\{0\}$, $\varphi(L)$ is defined. Since $\zeta\cdot\frac{1}{\zeta} = \frac{1}{\zeta}\cdot\zeta = 1$, $L\varphi(L) = \varphi(L)L = I$. Thus $\varphi(L) = L^{-1}$, as expected.

Similarly, one can show that if

$$\varphi(\zeta) = \frac{p(\zeta)}{q(\zeta)}$$

is a rational function (p, q are polynomials) and $\sigma(L) \subset \{\zeta : q(\zeta) \neq 0\}$, then

$$\varphi(L) = p(L)q(L)^{-1},$$

as expected.

To study our last operational property (composition), we need to identify $\sigma(\varphi(L))$.

The Spectral Mapping Theorem

Suppose $L \in \mathcal{L}(V)$ and φ is holomorphic in a neighborhood of $\sigma(L)$ (so $\varphi(L)$ is well-defined). Then

$$\sigma(\varphi(L)) = \varphi(\sigma(L)) \quad \textit{including multiplicities,}$$

i.e., if $\mu_1, \ldots, \mu_n$ are the eigenvalues of L counting multiplicities, then $\varphi(\mu_1), \ldots, \varphi(\mu_n)$ are the eigenvalues of $\varphi(L)$ counting multiplicities.

Proof. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of L, with algebraic multiplicities

$$m_1, \ldots, m_k,$$

respectively. By the residue theorem,

$$\begin{aligned}\varphi(L) &= -\frac{1}{2\pi i}\int_{\partial\Delta}\varphi(\zeta)R(\zeta)d\zeta \\ &= -\sum_{i=1}^{k}\mathcal{R}es_{\zeta=\lambda_i}[\varphi(\zeta)R(\zeta)].\end{aligned}$$

By the partial fractions decomposition of the resolvent,

$$-R(\zeta) = \sum_{i=1}^{k}\left(\frac{P_i}{\zeta-\lambda_i} + \sum_{\ell=1}^{m_i-1}(\zeta-\lambda_i)^{-\ell-1}N_i^{\ell}\right).$$

It follows that

$$\begin{aligned} -\mathcal{R}es_{\zeta=\lambda_i}\varphi(\zeta)R(\zeta) &= \varphi(\lambda_i)P_i + \sum_{\ell=1}^{m_i-1} \mathcal{R}es_{\zeta=\lambda_i}[\varphi(\zeta)(\zeta-\lambda_i)^{-\ell-1}]N_i^\ell \\ &= \varphi(\lambda_i)P_i + \sum_{\ell=1}^{m_i-1} \frac{1}{\ell!}\varphi^{(\ell)}(\lambda_i)N_i^\ell. \end{aligned}$$

Thus

$$(*) \qquad \varphi(L) = \sum_{i=1}^{k}[\varphi(\lambda_i)P_i + \sum_{\ell=1}^{m_i-1} \frac{1}{\ell!}\varphi^{(\ell)}(\lambda_i)N_i^\ell]$$

This is an explicit formula for $\varphi(L)$ in terms of the spectral decomposition of L and the values of φ and its derivatives at the eigenvalues of L. (In fact, this could have been used to define $\varphi(L)$, but our definition in terms of contour integration has the advantage that it generalizes to the infinite-dimensional case.) Since

$$\sum_{\ell=1}^{m_i-1} \frac{1}{\ell!}\varphi^{(\ell)}(\lambda_n)N_i^\ell$$

is nilpotent for each i, it follows that $(*)$ is the (unique) spectral decomposition of $\varphi(L)$. Thus

$$\sigma(\varphi(L)) = \{\varphi(\lambda_1), \ldots, \varphi(\lambda_k)\} = \varphi(\sigma(L)).$$

Moreover, if $\{\varphi(\lambda_1), \ldots, \varphi(\lambda_k)\}$ are distinct, then the algebraic multiplicity of $\varphi(\lambda_i)$ as an eigenvalue of $\varphi(L)$ is the same as that of λ_i for L, and they have the same eigenprojection P_i. In general, one must add the algebraic multiplicities and eigenprojections over all those i with the same $\varphi(\lambda_i)$. □

Remarks.

(1) The special case in which L is diagonalizable is easy to remember:

$$\text{if} \quad L = \sum_{i=1}^{k} \lambda_i P_i, \quad \text{then} \quad \varphi(L) = \sum_{i=1}^{k} \varphi(\lambda_i)P_i.$$

(2) Other consequences of $(*)$ are

$$\operatorname{tr}\varphi(L) = \sum_{i=1}^{k} m_i\varphi(\lambda_i) \quad \text{and} \quad \det\varphi(L) = \prod_{i=1}^{k} \varphi(\lambda_i)^{m_i}.$$

We now study composition.

Proposition. Suppose $L \in \mathcal{L}(V)$, φ_1 is holomorphic in a neighborhood of $\sigma(L)$, and φ_2 is holomorphic in a neighborhood of $\sigma(\varphi_1(L)) = \varphi_1(\sigma(L))$ (so $\varphi_2 \circ \varphi_1$ is holomorphic in a neighborhood of $\sigma(L)$). Then

$$(\varphi_2 \circ \varphi_1)(L) = \varphi_2(\varphi_1(L)).$$

Proof. Let Δ_2 contain $\sigma(\varphi_1(L))$ and let $\gamma_2 = \partial\Delta_2$. Then

$$\varphi_2(\varphi_1(L)) = \frac{1}{2\pi i}\int_{\gamma_2} \varphi_2(\zeta_2)(\zeta_2 I - \varphi_1(L))^{-1} d\zeta_2.$$

Here, $(\zeta_2 I - \varphi_1(L))^{-1}$ means of course the inverse of $\zeta_2 I - \varphi_1(L)$. For fixed $\zeta_2 \in \gamma_2$, we can also apply the functional calculus to the function $(\zeta_2 - \varphi_1(\zeta_1))^{-1}$ of ζ_1 to define this function of L: let Δ_1 contain $\sigma(L)$ and suppose that $\varphi_1(\bar{\Delta}_1) \subset \Delta_2$; then since $\zeta_2 \in \gamma_2$ is outside $\varphi_1(\bar{\Delta}_1)$, the map

$$\zeta_1 \mapsto (\zeta - 2 - \varphi_1(\zeta_1))^{-1}$$

is holomorphic in a neighborhood of $\bar{\Delta}_1$, so we can evaluate this function of L; just as for

$$\zeta \mapsto \frac{1}{\zeta}$$

in the example above, we obtain the usual inverse of $\zeta_2 - \varphi_1(L)$. So

$$(\zeta_2 - \varphi_1(L))^{-1} = -\frac{1}{2\pi i}\int_{\gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1.$$

Hence

$$\begin{aligned}
\varphi_2(\varphi_1(L)) &= -\frac{1}{(2\pi i)^2}\int_{\gamma_2} \varphi_2(\zeta_2)\int_{\gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1 d\zeta_2 \\
&= -\frac{1}{(2\pi i)^2}\int_{\gamma_1} R(\zeta_1)\int_{\gamma_2} \frac{\varphi_2(\zeta_2)}{\zeta_2 - \varphi_1(\zeta_1)} d\zeta_2 d\zeta_1 \\
&= -\frac{1}{2\pi i}\int_{\gamma_1} R(\zeta_1)\varphi_2(\varphi_1(\zeta_1)) d\zeta_1 \qquad \text{(as } n(\gamma_2, \varphi_1(\zeta_1)) = 1\text{)} \\
&= (\varphi_2 \circ \varphi_1)(L).
\end{aligned}$$

□

Logarithms of Invertible Matrices

As an application, let $L \in \mathcal{L}(V)$ be invertible. We can choose a branch of $\log\zeta$ which is holomorphic in a neighborhood of $\sigma(L)$ and we can choose an appropriate Δ in which $\log\zeta$ is defined, so we can form

$$\log L = -\frac{1}{2\pi i}\int_{\gamma} \log\zeta R(\zeta) d\zeta \quad \text{(where } \gamma = \partial\Delta\text{)}.$$

This definition will of course depend on the particular branch chosen, but since $e^{\log\zeta} = \zeta$ for any such branch, it follows that for any such choice,

$$e^{\log L} = L.$$

In particular, *every* invertible matrix is in the range of the exponential. This definition of the logarithm of an operator is much better than one can do with series: one could define

$$\log(I + A) = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{A^\ell}{\ell},$$

but the series only converges absolutely in norm for a restricted class of A, namely $\{A : \rho(A) < 1\}$.