# Norms on Operators

If $V$, $W$ are vector spaces then so is the space of linear transformations from $V$ to $W$ denoted $\mathcal{L}(V,W)$. We now consider norms on $\mathcal{L}(V,W)$. When $V=W$, $\mathcal{L}(V,V)=\mathcal{L}(V)$ is an algebra with composition as multiplication; norms on $\mathcal{L}(V)$ which have a relationship to composition are particularly useful. A norm on $\mathcal{L}(V)$ is said to be *submultiplicative* if $\|A\circ B\|\leq\|A\|\cdot\|B\|$. (H-J calls this a matrix norm in finite dimensions.)

*Example.* For $A\in\mathbb{C}^{n\times n}$, define $\|A\|=\sup_{1\leq i,j\leq n}|a_{ij}|$. This norm is not submultiplicative: if $A=B=\begin{pmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{pmatrix}$, then $\|A\|=\|B\|=1$, but $AB=A^2=nA$ so $\|AB\|=n$.

*Exercise.* Show that although the norm $\|A\|=\sup_{1\leq i,j\leq n}|a_{ij}|$ on $\mathbb{C}^{n\times n}$ is not multiplicative, the norm $A\mapsto n\sup_{1\leq i,j\leq n}|a_{ij}|$ is submultiplicative.

## Bounded Linear Operators and Operator Norms

Let $(V,\|\cdot\|_v)$ and $(W,\|\cdot\|_w)$ be normed linear spaces. An $L\in\mathcal{L}(V,W)$ is called a *bounded linear operator* if $\sup_{\|v\|_v=1}\|Lv\|_w<\infty$. Let $\mathcal{B}(V,W)$ denote the set of all bounded linear operators from $V$ to $W$. In the special case $W=\mathbb{F}$ we have *bounded linear functionals*, and we set $V^*=\mathcal{B}(V,\mathbb{F})$. If $\dim V<\infty$, then $\mathcal{L}(V,W)=\mathcal{B}(V,W)$, so also $V^*=V'$. In fact, if we choose a basis $\{v_1,\ldots,v_n\}$ for $V$ and let $\{f_1,\ldots,f_n\}$ be the dual basis, then $\sum_{i=1}^n|f_i(v)|$ is a norm on $V$ (see exercise below), so by the Norm Equivalence Theorem, $\exists\, M\ni\sum_{i=1}^n|f_i(v)|\leq M\|v\|$; then

$$\begin{aligned}\|Lv\|_w &= \left\|L\left(\sum_{i=1}^n f_i(v)v_i\right)\right\|_w\\ &\leq \sum_{i=1}^n|f_i(v)|\cdot\|Lv_i\|_w\\ &\leq \left(\max_{1\leq i\leq n}\|Lv_i\|_w\right)\sum_{i=1}^n|f_i(v)|\\ &\leq \left(\max_{1\leq i\leq n}\|Lv_i\|_w\right)M\|v\|_v,\end{aligned}$$

so

$$\sup_{v\neq0}\left(\|Lv\|_w/\|v\|_v\right)\leq\left(\max_{1\leq i\leq n}\|Lv_i\|_w\right)\cdot M<\infty.$$

(Recall that if $v=\sum_{i=1}^n x_iv_i$, then $x_i=f_i(v)$.)

*Caution.* A bounded linear operator doesn't necessarily have $\{\|Lv\|_w : v\in V\}$ being a bounded set of $\mathbb{R}$: in fact, if it is, then $L\equiv0$. Similarly, if a linear functional is a bounded linear functional, it does *not* mean that there is an $M$ for which $(\forall\, v\in V)\ |f(v)|\leq M$.

*Exercise.*

(1) Suppose $V$ is a finite dimensional vector space and let $\{v_1, \ldots, v_n\}$ be a basis for $V$ with associated dual basis $\{f_1, \ldots, f_n\}$. Show that the mapping $v \mapsto \sum_{i=1}^n |f_i(v)|$ defines a norm on $V$.

(2) Let $L \in \mathcal{L}(V, W)$ and show that $\sup_{\|v\|_v=1} \|Lv\|_w = \sup_{\|v\|_v \le 1} \|Lv\|_w = \sup_{v \neq 0}(\|Lv\|_w / \|v\|_v)$.

*Examples.*

(1) Let $V = \mathcal{P}$ be the space of polynomials with norm $\|p\| = \sup_{0 \le x \le 1} |p(x)|$. The differentiation operator $\frac{d}{dx} : \mathcal{P} \to \mathcal{P}$ is not a bounded linear operator: $\|x^n\| = 1$ for all $n \geq 1$¡ but $\left\|\frac{d}{dx} x^n\right\| = \|nx^{n-1}\| = n$.

(2) Let $V = \mathbb{F}_0^\infty$ with $\ell^p$-norm for some $p$, $1 \le p \le \infty$. Let $L$ be diagonal, so $Lx = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots)^T$ for $x \in \mathbb{F}_0^\infty$, where $\lambda_i \in \mathbb{C}$, $i \geq 1$. Then $L$ is a bounded linear operator iff $\sup_i |\lambda_i| < \infty$.

*Exercise.* Verify the claim in example (2) above.

We have already proved:

**Proposition.** Let $L : V \to W$ be a linear transformation between normed vector spaces. Then $L$ is bounded iff $L$ is continuous iff $L$ is uniformly continuous.

**Definition.** Let $L : V \to W$ be a bounded linear operator between normed linear spaces, i.e., $L \in \mathcal{B}(V, W)$. Define the operator norm of $L$ to be

$$\|L\| = \sup_{\|v\|_v \le 1} \|Lv\|_w \left( = \sup_{\|v\|_v = 1} \|Lv\|_w = \sup_{v \neq 0} (\|Lv\|_w / \|v\|_v) \right).$$

*Remark.* $(\forall\, v \in V)$ $\|Lv\|_w \le \|L\| \cdot \|v\|_v$. In fact, $\|L\|$ is the smallest constant with this property: $\|L\| = \min\{C \geq 0 : (\forall\, v \in V)\ \|Lv\|_w \le C\|v\|_v\}$.

We can now show that $\mathcal{B}(V, W)$ is a vector space (a subspace of $\mathcal{L}(V, W)$). If $L \in \mathcal{B}(V, W)$ and $\alpha \in \mathbb{F}$, clearly $\alpha L \in \mathcal{B}(V, W)$ and $\|\alpha L\| = |\alpha| \cdot \|L\|$. If $L_1, L_2 \in \mathcal{B}(V, W)$, then $\|(L_1 + L_2)v\|_w \le \|L_1 v\|_w + \|L_2 v\|_w \le (\|L_1\| + \|L_2\|)\|v\|_v$, so $L_1 + L_2 \in \mathcal{B}(V, W)$, and $\|L_1 + L_2\| \le \|L_1\| + \|L_2\|$. It follows that the operator norm is indeed a norm on $\mathcal{B}(V, W)$. $\|\cdot\|$ is sometimes called the operator norm on $\mathcal{B}(V, W)$ induced by the norms $\|\cdot\|_v$ and $\|\cdot\|_w$ (as it clearly depends on both $\|\cdot\|_v$ and $\|\cdot\|_w$).

In the special case $W = \mathbb{F}$, the norm $\|f\| = \sup_{\|v\| \le 1} |f(v)|$ on $V^*$ is called the *dual norm* to that on $V$. If $\dim V < \infty$, then we can choose bases and identify $V$ and $V^*$ with $\mathbb{F}^n$. Thus every norm on $\mathbb{F}^n$ has a dual norm on $\mathbb{F}^n$. We sometimes write $F^{n^*}$ for $\mathbb{F}^n$ when it is being identified with $V^*$. Consider some examples.

*Examples.*

(1) If $\mathbb{F}^n$ is given the $\ell'$-norm, then the dual norm is $\|f\| = \max_{\|x\|_1 \le 1} \left|\sum_{i=1}^n f_i x_i\right|$ for $f = (f_1, \ldots, f_n) \in \mathbb{F}^{n^*}$, which is easily seen to be the $\ell^\infty$-norm $\|f\|_\infty$ (exercise).

(2) If $\mathbb{F}^n$ is given the $\ell^\infty$-norm, then the dual norm is $\|f\| = \max_{\|x\|_\infty \le 1} |\sum_{i=1}^n f_i x_i|$ for $f = (f_1, \ldots, f_n) \in \mathbb{F}^{n^*}$, which is easily seen to be the $\ell'$-norm $\|f\|_1$ (exercise).

(3) The dual norm to the $\ell^2$-norm on $\mathbb{F}^n$ is again the $\ell^2$-norm; this follows easily from the Schwarz inequality (exercise). The $\ell^2$-norm is the only norm on $\mathbb{F}^n$ which equals its own dual norm; see the homework.

(4) Let $1 < p < \infty$. The dual norm to the $\ell^p$-norm on $\mathbb{F}^n$ is the $\ell^q$-norm, where $\frac{1}{p} + \frac{1}{q} = 1$¿ The key inequality is Hölder's inequality: $|\sum_{i=1}^n f_i x_i| \le \|f\|_q \cdot \|x\|_p$. We will be primarily interested in the cases $p = 1, 2, \infty$. (Note: $\frac{1}{p} + \frac{1}{q} = 1$ in an extended sense when $p = 1$ and $q = \infty$, or when $p = \infty$ and $q = 1$; Hölder's inequality is trivial in these cases.)

It is instructive to consider linear functionals and the dual norm geometrically. Recall that a norm on $\mathbb{F}^n$ can be described geometrically by its closed unit ball $B$, a compact convex set. The geometric realization of a linear functional (excluding the zero functional) is a hyperplane. (A hyperplane in $\mathbb{F}^n$ is a set of the form $\{x \in \mathbb{F}^n : \sum_{i=1}^n f_i x_i = c\}$, where $f_i$, $i \in \mathbb{F}$ and not all $f_i = 0$; sets of this form are sometimes called *affine* hyperplanes if the term "hyperplane" is being reserved for a subspace of $\mathbb{F}^n$ of dimension $n-1$.) In fact, there is a natural $1-1$ correspondence between $\mathbb{F}^{n^*}\backslash\{0\}$ and the set of hyperplanes in $\mathbb{F}^n$ which do not contain the origin: to $f = (f_1, \ldots, f_n) \in \mathbb{F}^{n^*}$, associate the hyperplane $\{x \in \mathbb{F}^n : f(x) = f_1 x_1 + \cdots + f_n x_n = 1\}$; since every hyperplane not containing 0 has a unique equation of this form, this is a $1-1$ correspondence as claimed.

If $\mathbb{F} = \mathbb{C}$ it is often more appropriate to use real hyperplanes in $\mathbb{C}^n = \mathbb{R}^{2n}$; if $z \in \mathbb{C}^n$ and we write $z_j = x_j + iy_j$, then a real hyperplane not containing $\{0\}$ has a unique equation of the form $\sum_{j=1}^n (a_j x_j + b_j y_j) = 1$ where $a_j, b_j \in \mathbb{R}$, and not all of the $a_j$'s and $b_j$'s vanish. Observe that this equation is of the form $\mathcal{R}e\left(\sum_{j=1}^n f_j z_j\right) = 1$ where $f_j = a_j - ib_j$ is uniquely determined. Thus the real hyperplanes in $\mathbb{C}^n$ not containing $\{0\}$ are all of the form $\mathcal{R}ef(z) = 1$ for a unique $f \in \mathbb{C}^{n^*}\backslash\{0\}$.

**Proposition.** If $(V, \|\cdot\|)$ is a normed linear space and $f \in V^*$, then the dual norm of $f$ satisfies $\|f\| = \sup_{\|v\| \le 1} \mathcal{R}ef(v)$.

**Proof.** Since $\mathcal{R}ef(v) \le |f(v)|$, $\sup_{\|v\| \le 1} \mathcal{R}ef(v) \le \sup_{\|v\| \le 1} |f(v)| = \|f\|$. For the other direction, choose a sequence $\{v_j\}$ from $V$ with $\|v_j\| = 1$ and $|f(v_j)| \to \|f\|$. Taking $\theta_j = -\arg f(v_j)$ and setting $w_j = e^{i\theta_j} v_j$, we have $\|w_j\| = 1$ and $f(w_j) = |f(v_j)| \to \|f\|$, so $\sup_{\|v\| \le 1} \mathcal{R}ef(v) \ge \|f\|$. □

With these observations, we can give a description of the dual unit ball in terms of the geometry of the hyperplanes and the unit ball in the original norm. By the above, $f \in \mathbb{F}^{n^*}$ satisfies $\|f\| \le 1$ iff $\sup_{\|v\| \le 1} \mathcal{R}ef(v) \le 1$, i.e., iff the unit ball $B \subset \mathbb{F}^n$ is contained in the closed half-space $\mathcal{R}ef(v) \le 1$ (the real hyperplane $\{\mathcal{R}ef(v) = 1\}$ divides $\mathbb{F}^n$ into two half-spaces; this is the one containing the origin). Moreover, by linearity, if $\|f\| \le 1$ and $\|v\| = p < 1$, then $\mathcal{R}ef(v) \le p < 1$, so the open unit ball $B^0 \subset \{f : \mathcal{R}ef(v) < 1 \ \forall\, v \in B\}$. So we have a description of the dual unit ball on those functionals corresponding to hyperplanes lying outside the open unit ball $B^0 = \{f : \|f\| < 1\}$. It is interesting to translate this into a geometric dual unit ball in specific examples; see the homework.

**Proposition.** If $(V, \|\cdot\|)$ is a normed linear space and $v \in V$, then

$$(\exists\, f \in V^*) \ni \|f\| = 1 \qquad \text{and} \qquad f(v) = \|v\|.$$

In general, this is an immediate consequence of the Hahn-Banach theorem (see, e.g., Royden *Real Analysis* or Folland *Real Analysis*), and for convenience we will refer to it here as the Hahn-Banach theorem. In finite dimensions, there are more geometric proofs based on relating hyperplanes to the closed unit ball. See, e.g., Corollary 5.5.15 in H-J (see also Appendix B in H-J).

## Consequences of the Hahn-Banach theorem

### The Second Dual

Let $(V, \|\cdot\|)$ be a normed linear space, $V^*$ be its dual equipped with the dual norm, and $V^{**}$ be the dual of $V^*$ with the norm dual to that on $V^*$. Given $v \in V$, define $v^{**} \in V^{**}$ by $v^{**}(f) = f(v)$; since $|v^{**}(f)| \leq \|f\| \cdot \|v\|$, $v^{**} \in V^{**}$ and $\|v^{**}\| \leq \|v\|$. By the Hahn-Banach theorem, $\exists\, f \in V^*$ with $\|f\| = 1$ and $f(v) = \|v\|$, i.e., $v^{**}(f) = \|v\|$, so $\|v^{**}\| = \sup_{\|f\|=1} |v^{**}(f)| \geq \|v\|$. Hence $\|v^{**}\| = \|v\|$, so the mapping $v \mapsto v^{**}$ from $V$ into $V^{**}$ is an isometry of $V$ onto the range of this map. In general, this embedding is not surjective; if it is, then $(V, \|\cdot\|)$ is called *reflexive*

In finite dimensions, dimension arguments imply this map is surjective. Thus the dual norm to the dual norm is just the original norm on $V$.

### Adjoint Transformations

Recall that if $L \in \mathcal{L}(V, W)$, the adjoint transformation $L^* : W' \to V'$ is given by $(L^*g)(v) = g(Lv)$.

**Proposition.** Let $V$, $W$ be normed linear spaces. If $L \in \mathcal{B}(V, W)$, then $L^*[W^*] \subset V^*$. Moreover, $L^* \in \mathcal{B}(W^*, V^*)$ and $\|L^*\| = \|L\|$.

**Proof.** For $g \in W^*$, $|(L^*g)(v)| = |g(Lv)| \leq \|g\| \cdot \|L\| \cdot \|v\|$, so $L^*g \in V^*$, and $\|L^*g\| \leq \|g\| \cdot \|L\|$. Thus $L^* \in \mathcal{B}(W^*, V^*)$ and $\|L^*\| \leq \|L\|$. Now given $v \in V$, apply the Hahn-Banach theorem to $Lv$ to conclude that $\exists\, g \in W^*$ with $\|g\| = 1$ and $(L^*g)(v) = g(Lv) = \|Lv\|$. So $\|L^*\| = \sup_{\|g\|\leq 1} \|L^*g\| = \sup_{\|g\|\leq 1} \sup_{\|v\|\leq 1} |(L^*g)(v)| \geq \sup_{\|v\|\leq 1} \|Lv\| = \|L\|$. Hence $\|L^*\| = \|L\|$. $\square$

## Completeness of $\mathcal{B}(V, W)$ when $W$ is complete

**Proposition.** If $W$ is complete, the $\mathcal{B}(V, W)$ is complete. In particular, $V^*$ is always complete (since $\mathbb{F}$ is), whether or not $V$ is.

**Proof.** If $\{L_n\}$ is Cauchy in $\mathcal{B}(V, W)$, then $(\forall\, v \in V)\{L_n v\}$ is Cauchy in $W$, so the limit $\lim_{n\to\infty} L_n v \equiv Lv$ exists in $W$. Clearly $L : V \to W$ is linear, and it is easy to see that $L \in \mathcal{B}(V, W)$ and $\|L_n - L\| \to 0$. $\square$

## Analysis with Operators

Throughout this discussion, let $V$ be a Banach space. Since $V$ is complete, $\mathcal{B}(V) = \mathcal{B}(V, V)$ is also complete (in the operator norm).

**Fact.** Operator norms are always submultiplicative.

In fact, if $U, V, W$ are normed linear spaces and $L \in \mathcal{B}(U, V)$ and $M \in \mathcal{B}(V, W)$, then for $u \in U$,

$$\|(M \circ L)(u)\|_W = \|M(Lu)\|_W \leq \|M\| \cdot \|Lu\|_V \leq \|M\| \cdot \|L\| \cdot \|u\|_u,$$

so $M \circ L \in \mathcal{B}(U, W)$ and $\|M \circ L\| \leq \|M\| \cdot \|L\|$. The special case $U = V = W$ shows that the operator norm on $\mathcal{B}(V)$ is submultiplicative (and $L, M \in \mathcal{B}(V) \Rightarrow M \circ L \in \mathcal{B}(V)$). We want to define functions of an operator $L \in \mathcal{B}(V)$. We can compose $L$ with itself, so we can form powers $L^k = L \circ \cdots \circ L$, and thus we can define polynomials in $L$: if $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then $p(L) \equiv a_0 I + a_1 L + \cdots + a_n L^n$. By taking limits, we can form power series, and thus analytic functions of $L$. For example, consider the series $e^L = \sum_{k=0}^{\infty} \frac{1}{k!} L^k = I + L + \frac{1}{2} L^2 + \cdots$ (note $L^0$ is the identity $I$ by definition). This series converges in the operator norm on $\mathcal{B}(V)$: by submultiplicativity, $\|L^k\| \leq \|L\|^k$, so $\sum_{k=0}^{\infty} \frac{1}{k!} \|L^k\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|L\|^k = e^{\|L\|} < \infty$; since the series converges absolutely and $\mathcal{B}(V)$ is complete (recall $V$ is a Banach space), it converges in the operator norm to an operator in $\mathcal{B}(V)$ which we call $e^L$ (note that $\|e^L\| \leq e^{\|L\|}$). In the finite dimensional case, this says that for $A \in \mathbb{F}^{n \times n}$, each component of the partial sum $\sum_{k=0}^{N} \frac{1}{k!} A^k$ converges as $N \to \infty$; the limiting matrix is $e^A$.

Another fundamental example is the Neumann series.

**Proposition.** If $L \in \mathcal{B}(V)$ and $\|L\| < 1$, then $I - L$ is invertible, and the Neumann series $\sum_{k=0}^{\infty} L^k$ converges in the operator norm to $(I - L)^{-1}$.

*Remark.* Formally we can guess this result since the power series of $\frac{1}{1-z}$ centered at $z = 0$ is $\sum_{k=0}^{\infty} z^k$ with radius of convergence 1.

**Proof.** If $\|L\| < 1$, then $\sum_{k=0}^{\infty} \|L^k\| \leq \sum_{k=0}^{\infty} \|L\|^k = \frac{1}{1 - \|L\|} < \infty$, so the Neumann series $\sum_{k=0}^{\infty} L^k$ converges to an operator in $\mathcal{B}(V)$. Now if $S_j, S, T \in \mathcal{B}(V)$ and $S_j \to S$ in $\mathcal{B}(V)$, then $\|S_j - S\| \to 0$, so $\|S_j T - ST\| \leq \|S_j - S\| \cdot \|T\| \to 0$ and $\|T S_j - TS\| \leq \|T\| \cdot \|S_j - S\| \to 0$, and thus $S_j T \to ST$ and $T S_j \to TS$ in $\mathcal{B}(V)$. Thus $(I - L) \left( \sum_{k=0}^{\infty} L^k \right) = \lim_{N \to \infty} (I - L) \sum_{k=0}^{N} L^k = \lim_{N \to \infty} (I - L^{N+1}) = I$ (as $\|L^{N+1}\| \leq \|L\|^{N+1} \to 0$), and similarly $\left( \sum_{k=0}^{\infty} L^k \right) (I - L) = I$. So $I - L$ is invertible and $(I - L)^{-1} = \sum_{k=0}^{\infty} L^k$. □

This is a very useful fact: a perturbation of $I$ by an operator of norm $< 1$ is invertible. This implies, among other things, that the set of invertible operators in $\mathcal{B}(V)$ is an open subset of $\mathcal{B}(V)$ (in the operator norm).

Our terminology above is that an operator in $\mathcal{B}(V)$ is called invertible if it is bijective (i.e., invertible as a point-set mapping from $V$ onto $V$, which implies that the inverse map is well-defined and linear) *and* that its inverse is also in $\mathcal{B}(V)$.

*Note*: $\mathcal{B}(V)$ has a ring structure using the addition of operators, and composition of operators as the multiplication; the identity of multiplication is just the identity operator $I$. Our concept of invertibility is equivalent to invertibility in this ring: if $L \in \mathcal{B}(V)$ and $\exists\, M \in$

$\mathcal{B}(V) \ni LM = ML = I$, then $ML = I \Rightarrow L$ injective and $LM = I \Rightarrow L$ surjective. Note that this ring in general is *not* commutative.

It is a consequence of the closed graph theorem (see Royden or Folland) that if $L \in \mathcal{B}(V)$ is bijective (and $V$ is a Banach space), then its inverse map $L^{-1}$ is also in $\mathcal{B}(V)$.

Clearly the power series arguments used above can be generalized. Let $f(z)$ be analytic on the disk $\{|z| < R\} \subset \mathbb{C}$, with power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ (which has radius of convergence at least $R$). If $L \in \mathcal{B}(V)$ and $\|L\| < R$, then the series $\sum_{k=0}^{\infty} a_k L^k$ converges absolutely, and thus converges to an element of $\mathcal{B}(V)$ which we call $f(L)$ (recall $V$ is a Banach space). It is easy to check that usual operational properties hold, for example $(fg)(L) = f(L)g(L) = g(L)f(L)$. However, one must be careful to remember that operators do not commute in general. So, for example, $e^{L+M} \neq e^L e^M$ in general, although if $L$ and $M$ commute (i.e. $LM = ML$), then $e^{L+M} = e^L e^M$.

Let $L(t)$ be a 1-parameter family of operators in $\mathcal{B}(V)$, where $t \in (a,b)$. Since $\mathcal{B}(V)$ is a metric space, we know what it means for $L(t)$ to be a continuous function of $t$. We can define differentiability as well: $L(t)$ is differentiable at $t = t_0 \in (a,b)$ if $L'(t_0) = \lim_{t\to t_0} \frac{L(t)-L(t_0)}{t-t_0}$ exists in the norm on $\mathcal{B}(V)$. For example, it is easily checked that for $L \in \mathcal{B}(V)$, $e^{tL}$ is differentiable in $t$ for all $t \in \mathbb{R}$, and $\frac{d}{dtd} e^{tL} = Le^{tL} = e^{tL}L$.

We can similarly consider families of operators in $\mathcal{B}(V)$ depending on several real parameters or on complex parameter(s). A family $L(z)$ where $z = x + iy \in \Omega^{\text{open}} \subset \mathbb{C}$ $(x, y \in \mathbb{R})$ is said to be holomorphic in $\Omega$ if the partial derivatives $\frac{\partial}{\partial x} L(z)$, $\frac{\partial}{\partial y} L(z)$ exist and are continuous in $\Omega$, and $L(z)$ satisfies the Cauchy-Riemann equation $\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) L(z) = 0$ in $\Omega$. As in complex analysis, this is equivalent to the assumption that in a neighborhood of each point $z_0 \in \Omega$, $L(z)$ is given by the $\mathcal{B}(V)$-norm convergent power series $L(z) = \sum_{k=0}^{\infty} \frac{1}{k!}(z - z_0)^k \left(\frac{d}{dz}\right)^k L(z_0)$.

One can also integrate families of operators. If $L(t)$ depends continuously on $t \in [a,b]$, then it can be shown using the same estimates as for $\mathbb{F}$-valued functions (and the uniform continuity of $L(t)$ since $[a,b]$ is compact) that the Riemann sums $\frac{b-a}{N} \sum_{k=0}^{n-1} L\left(a + \frac{k}{N}(b-a)\right)$ converge in $\mathcal{B}(V)$-norm (recall $V$ is a Banach space) as $n \to \infty$ to an operator in $\mathcal{B}(V)$, denoted $\int_a^b L(t)dt$. (More general Riemann sums than just the left-hand "rectangular rule" with equally spaced points can be used.) Many results from standard calculus carry over, including $\left\|\int_a^b L(t)dt\right\| \leq \int_a^b \|L(t)\|dt$ which follows directly from $\left\|\frac{b-a}{N} \sum_{k=0}^{N-1} L\left(a + \frac{k}{N}(b-a)\right)\right\| \leq \frac{b-a}{N} \sum_{k=0}^{N-1} \left\|L\left(a + \frac{k}{N}(b-a)\right)\right\|$. By parameterizing paths in $\mathbb{C}$, one can define line integrals of holomorphic families of operators. We will discuss such constructions further as we need them.

## Operators in Finite Dimensions

In the next part of the course we will study in greater detail operators in finite dimensions and the matrices which represent them.

## Transposes and Adjoints

If $A \in \mathbb{C}^{m\times n}$ we denote by $A^T \in \mathbb{C}^{n\times m}$ the transpose of $A$, and by $A^H = \bar{A}^T$ the conjugate-transpose (or hermitian transpose) of $A$. (Many books, including H-J, use the notation $A^*$ for $A^H$.) If $x, y \in \mathbb{C}^n$ are represented in terms of matrix multiplication as $\langle x, y\rangle = y^H x$. For $A \in \mathbb{C}^{n\times n}$, we then have $\langle Ax, y\rangle = \langle x, A^H y\rangle$ since $y^H Ax = (A^H y)^H x$.

*Caution:* The notation $A^*$, or $L^*$ for a linear transformation, is used with two different, sometimes contradictory meaning, particularly if $\mathbb{F} = \mathbb{C}$. Recall that if $L \in \mathcal{B}(V, W)$ then $L^* \in \mathcal{B}(W^*, V^*)$ and in the finite dimensional case, we saw that if $L$ corresponds to matrix multiplication on column vectors from the left by the matrix $T$, then $L^*$ corresponds to matrix multiplication on row vectors from the right by the matrix $T$, or equivalently by transposition to left-multiplication by the transpose matrix $T^T$ on column vectors. On the other hand, in the presence of an inner product, the usual definition $\langle Lx, y\rangle = \langle x, L^* y\rangle$ identifies $L^*$ with left-multiplication by the conjugate-transpose matrix. These two definitions are related by the identification $V \cong V^*$ induced by the inner product, but the conjugation in this identification gives rise to the two inequivalent definitions of $L^*$. So you must be careful to be sure which is meant in a given context. (Some authors use the notation $V'$ for $V^* = \mathcal{B}(V, \mathbb{F})$ and the notation $L' \in \mathcal{B}(W', V')$ for the transpose operator, reserving the notation $L^*$ for use with inner products.)

## Norms on Matrices

Commonly used norms on $\mathbb{C}^{n\times n}$ are the following. (We use the notation of H-J.)

$$\|A\|_1 = \sum_{i,j=1}^n |a_{ij}| \qquad \text{(the } \ell'\text{-norm on } A \text{ as if it were in } \mathbb{C}^{n^2}\text{)}$$

$$\|A\|_\infty = \max_{i,j} |a_{ij}| \qquad \text{(the } \ell^\infty\text{-norm on } A \text{ as if it were in } \mathbb{C}^{n^2}\text{)}$$

$$\|A\|_2 = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}} \qquad \text{(the } \ell^2\text{-norm on } A \text{ as if it were in } \mathbb{C}^{n^2}\text{)}$$

The norm $\|A\|_2$ is called the Hilbert-Schmidt norm of $A$, or the Frobenius norm of $A$, and is often denoted $\|A\|_F$. It is sometimes called the Euclidean norm of $A$. This norm comes from an inner product $\langle A, B\rangle = \sum_{i,j=1}^n a_{ij}\overline{b_{ij}} = \mathrm{tr}\,(B^* A)$.

We also have the following $p$-norms for matrices: let $1 \leq p \leq \infty$, then

$$|||A|||_p = \max_{\|x\|_p = 1} \|Ax\|_p \qquad \left( = \max_{\|x\|_p \leq 1} \|Ax\|_p = \max_{x\neq 0}(\|Ax\|_p / \|x\|_p)\right).$$

*Caution:* $|||A|||_p$ is a quite non-standard notation; the standard notation is $\|A\|_p$, and a more standard notation for the Frobenius norm is $\|A\|_F$, particularly in numerical analysis. We will, however, go ahead and use the notation of H-J.

Using arguments similar to those identifying the dual norms to the $\ell'$- and $\ell^\infty$-norms on $\mathbb{C}^n$, it can be easily shown that

$$|||A|||_1 = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}| \qquad \text{(maximum (absolute) column sum)}$$

$$|||A|||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}| \qquad \text{(maximum (absolute) row sum)}$$

$|||A|||_2$ is often called the spectral norm (we will show later that it equals the square root of the largest eigenvalues of $A^H A$.)

All of the above norms are submultiplicative except for $\|\cdot\|_\infty$, which we have previously discussed.

## Consistent Matrix Norms

The concept of submultiplicativity can be extended to rectangular matrices.

**Definition.** Let $\mu : \mathbb{C}^{m\times n} \to \mathbb{R}$, $\nu : \mathbb{C}^{n\times k} \to \mathbb{R}$, $\rho : \mathbb{C}^{m\times k} \to \mathbb{R}$ be norms. We say that $\mu, \nu, \rho$ are *consistent* if $\forall\, A \in \mathbb{C}^{m\times n}$ and $\forall\, B \in \mathbb{C}^{n\times k}$,

$$\rho(AB) \le \mu(A)\nu(B)$$

**Definition.** A norm on $\mathbb{F}^{n\times n}$ is called consistent if it is consistent with itself, i.e., the definition above with $m = n = k$ an $\rho = \mu = \nu$. So by definition a norm on $\mathbb{F}^{n\times n}$ is consistent iff it is submultiplicative.

In this discussion of consistent matrix norms, we identify $\mathbb{F}^n$ with $\mathbb{F}^{n\times 1}$ (i.e., $n\times 1$ matrices or column vectors).
*Examples.*

(1) Let $k = 1$. Then $\rho$ is a norm on $\mathbb{F}^m$ $(\cong \mathbb{F}^{m\times 1})$, $\nu$ is a norm on $\mathbb{F}^n$ $(\cong \mathbb{F}^{n\times 1})$, and $\mu$ is a norm on $\mathbb{F}^{m\times n}$. If $\mu_0$ is the operator norm induced by $\nu$ and $\rho$, then $\forall\, A \in \mathbb{F}^{m\times n}$ and $\forall\, x \in \mathbb{F}^n$, $\rho(Ax) \le \mu_0(A)\nu(x)$, so $\mu_0, \nu$, and $\rho$ are consistent.

(2) Again, let $k = 1$, and $\rho$ and $\nu$ be norms on $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively. Let $\mu$ be a norm on $\mathbb{F}^{m\times n}$. Then $\mu, \nu, \rho$ are consistent iff $\mu \ge \mu_0$ where $\mu_0$ is the operator norm on $\mathbb{F}^{m\times n}$ induced by $\nu$ and $\rho$. (For each $A \in \mathbb{F}^{m\times n}$, $(\forall\, x \in \mathbb{F}^n)$ $\rho(Ax) \le \mu(A)\nu(x)$ iff $(\forall\, x \ne 0)\rho(Ax)/\nu(x) \le \mu(A)$ iff $\mu_0(A) \le \mu(A)$.)

## Families of Matrix Norms

A collection $\{\nu_{m,n} : m \ge 1, n \ge 1\}$, where $\nu_{m,n} = \mathbb{F}^{m\times n} \to \mathbb{R}$ is a norm on $\mathbb{F}^{m\times n}$, is called a *family of matrix norms* (we temporarily discard the H-J assumption of submultiplicativity on the "matrix norms" $\nu_{n,n}$).

**Definition.** A family $\{\nu_{m,n} : m \ge 1, n \ge 1\}$ of matrix norms is called *consistent* if

$$(\forall\, m, n, k \ge 1)(\forall\, A \in \mathbb{F}^{m\times n})(\forall\, B \in \mathbb{F}^{n\times k})\ \nu_{m,k}(AB) \le \nu_{m,n}(A)\nu_{n,k}(B).$$

**Facts**: Let $\{\nu_{m,n}\}$ be a consistent family of matrix norms. Then

(1) $(\forall\, n \ge 1)$ $\nu_{n,n}$ is submultiplicative.

(2) $(\forall\, m, n \geq 1)$ $(\forall\, A \in \mathbb{F}^{m\times n})$ $\nu_{m,n}(A) \geq \mu_{m,n}(A)$, where $\mu_{m,n}$ is the operator norm on $\mathbb{F}^{m\times n}$ induced by $\nu_{n,1}$ and $\nu_{m,1}$.

*Examples.*

(1) For $m \geq 1$, let $\nu_{m,1}$ be any norm on $\mathbb{F}^m$. For $m, n \geq 1$, let $\nu_{m,n}$ be the operator norm on $\mathbb{F}^{m\times n}$ induced by $\nu_{n,1}$ and $\nu_{m,1}$ (to avoid contradicting definitions of $\nu_{m,1}$, we take $\nu_{1,1}$ to be the usual absolute value on $\mathbb{F}$). Then $\{\nu_{m,n}\}$ is a consistent family of matrix norms.

(2) (maximum (absolute) row sum norm) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let $\nu_{m,n}(A) = \max_{1\leq i\leq m} \sum_{j=1}^n |a_{ij}|$. Then $\nu_{n,1}$ is the $\ell^\infty$-norm on $\mathbb{F}^n$, and $\nu_{m,n}(A)$ is the operator norm induced by the $\ell^\infty$-norms on $\mathbb{F}^n$ and $\mathbb{F}^m$ (exercise), which we denote by $|||A|||_\infty$ (even for $m \neq n$). This is a special case of example (1), so it is a consistent family of matrix norms.

(3) (maximum (absolute) column sum norm) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let $\nu_{m,n}(A) = \max_{1\leq j\leq n} \sum_{i=1}^m |a_{ij}|$. Then $\nu_{n,1}$ is the $\ell'$-norm on $\mathbb{F}^n$, and $\nu_{m,n}(\cdot)$ is the operator norm induced by the $\ell'$-norms on $\mathbb{F}^n$ and $\mathbb{F}^m$ (exercise), which we denote by $|||A|||_1$ (even for $m \neq n$). This again is a special case of example (1), so it is a consistent family of matrix norms.

(4) ($\ell'$-norm on $\mathbb{F}^{m\times n}$ as if it were $\mathbb{F}^{mn}$) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let $\nu_{m,n}(A) = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|$. Then $\{\nu_{m,n}\}$ is a consistent family of matrix norms (exercise). We denote $\nu_{m,n}(A)$ by $\|A\|_1$ (even for $m \neq n$). Note that $\nu_{n,1}$ is the $\ell'$-norm on $\mathbb{F}^n$. This is *not* a special case of example (1). Note also that the obvious fact $\|A\|_1 \geq |||A|||_1$ agrees with Fact (2) above.

(5) ($\ell^2$-norm on $\mathbb{F}^{m\times n}$ as if it were $\mathbb{F}^{mn}$, i.e., the Hilbert-Schmidt or Frobenius norm) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let $\nu_{m,n}(A) = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}}$. Then $\nu_{n,1}$ is the $\ell^2$-norm on $\mathbb{F}^n$. If $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times k}$, then by the Schwarz inequality, $(\nu_{m,k}(AB))^2 = \sum_{i=1}^m \sum_{j=1}^k \left|\sum_{\ell=1}^n a_{i\ell} b_{\ell_j}\right|^2 \leq \sum_{i=1}^m \sum_{j=1}^n \left(\sum_{\ell=1}^n |a_{i\ell}|^2\right)\left(\sum_{r=1}^n |b_{rj}|^2\right) = (\nu_{m,n}(A)\nu_{n,k}(B))^2$, so $\{v_{m,n}\}$ is a consistent family of matrix norms. This is *not* a special case of example (1): for example, for $n > 1$, $\nu_{n,n}(I) = \sqrt{n}$ but the operator norm of $I$ is 1. We denote $\nu_{m,n}(A)$ by $\|A\|_2$ (even for $m \geq n$) (although most authors use $\|A\|_F$ for the Frobenius norm). For $A \in \mathbb{F}^{m\times n}$ and $x \in \mathbb{F}^n$, we have the inequality $\|Ax\|_2 \leq \|A\|_2 \cdot \|x\|_2$. For $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times k}$, $\|AB\|_2 \leq \|A\|_2 \cdot \|B\|_2$. Fact (2) above gives the important inequality: for $A \in \mathbb{F}^{m\times n}$, $|||A|||_2 \leq \|A\|_2$. Thus the operator norm induced by the $\ell^2$-norms on $\mathbb{F}^m$ and $\mathbb{F}^n$, which is not trivial to compute, is dominated by the Frobenius norm, which *is* easy to compute.

## Condition Number and Error Sensitivity

Throughout this discussion $A \in \mathbb{C}^{n\times n}$ will be assumed to be invertible. We are interested in determining the sensitivity of the solution of the linear system $Ax = b$ (for a given $b \in \mathbb{C}^n$) to perturbations in the right-hand-side (RHS) vector $b$ or to perturbations in $A$. One can think of such perturbations as arising from errors in measured data in computational problems, as often occurs when the entries in $A$ and/or $b$ are measured. As we will see, the fundamental quantity is the *condition number* $\kappa(A) = \|A\| \cdot \|A^{-1}\|$ of $A$, relative to a submultiplicative norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. Since $\|I\| \geq 1$ in any submultiplicative norm ($\|I\| = \|I^2\| \leq \|I\|^2 \Rightarrow \|I\| \geq 1$), $\kappa(A) = \|A\| \cdot \|A^{-1}\| \geq \|A \cdot A^{-1}\| = \|I\| \geq 1$.

Suppose $\|\cdot\|$ is a norm on $\mathbb{C}^{n\times n}$ consistent with a norm $\|\cdot\|$ on $\mathbb{C}^n$ (i.e. $\|Ax\| \leq \|A\|\cdot\|x\|$ as defined previously). Suppose first that the RHS vector $b$ is subject to error, but the matrix $A$ is not. Then one actually solves the system $A\widehat{x} = \widehat{b}$ for $\widehat{x}$, where $\widehat{b}$ is presumably close to $b$, instead of the system $Ax = b$ for $x$. Let $x$, $\widehat{x}$ be the solutions of $Ax = b$, $A\widehat{x} = \widehat{b}$, respectively. Define the *error vector* $e = x - \widehat{x}$, and the *residual vector* $r = b - \widehat{b} = b - A\widehat{x}$ (the amount by which $A\widehat{x}$ fails to match $b$). Then $Ae = A(x - \widehat{x}) = b - \widehat{b} = r$, so $e = A^{-1}r$. Thus $\|e\| \leq \|A^{-1}\| \cdot \|r\|$. Since $Ax = b$, $\|b\| \leq \|A\| \cdot \|x\|$. Multiplying these two inequalities gives $\|e\| \cdot \|b\| \leq \|A\| \cdot \|A^{-1}\| \cdot \|x\| \cdot \|r\|$, i.e. $\frac{\|e\|}{\|x\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}$. So the *relative error* $\frac{\|e\|}{\|x\|}$ is bounded by the condition number $\kappa(A)$ times the *relative residual* $\frac{\|r\|}{\|b\|}$.

*Exercise.* Let $A$, $b$, $x$, $e$, and $r$ be as given above and show that $\frac{\|e\|}{\|x\|} \geq \frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|}$.

Matrices for which $\kappa(A)$ is large are called *ill-conditioned* (relative to the norm $\|\cdot\|$); those for which $\kappa(A)$ is closed to $\|I\|$ (which is 1 if $\|\cdot\|$ is the operator norm) are called *well-conditioned* (and perfectly conditioned if $\kappa(A) = \|I\|$). If $A$ is ill-conditioned, small relative errors in the data (RHS vector $b$) *can* result in large relative errors in the solution.

If $\widehat{x}$ is the result of a numerical algorithm (with round-off error) for solving $Ax = b$, then the error $e = x - \widehat{x}$ is not computable, but the residual $r = b - A\widehat{x}$ is computable, so we obtain an upper bound on the relative error $\frac{\|e\|}{\|x\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}$. In practice, we don't know $\kappa(A)$ (although we may be able to estimate it), and this upper bound may be much larger than the actual relative error.

Suppose now that $A$ is subject to error, but $b$ is not. Then $\widehat{x}$ is now the solution of $(A+E)\widehat{x} = b$, where we assume that the error $E \in \mathbb{C}^{n\times n}$ in the matrix is small enough that $\|A^{-1}E\| < 1$, so $(I + A^{-1}E)^{-1}$ exists and can be computed by a Neumann series; then $A + E$ is invertible and $(A+E)^{-1} = (I + A^{-1}E)^{-1}A^{-1}$. The simplest inequality bounds $\frac{\|e\|}{\|\widehat{x}\|}$, the error relative to $\widehat{x}$, in terms of the relative error $\frac{\|E\|}{\|A\|}$ in $A$: the equations $Ax = b$ and $(A+E)\widehat{x} = b$ imply $A(x - \widehat{x}) = E\widehat{x}$, $x - \widehat{x} = A^{-1}E\widehat{x}$, and thus $\|x - \widehat{x}\| \leq \|A^{-1}\| \cdot \|E\| \cdot \|\widehat{x}\|$, so that

$$\frac{\|e\|}{\|\widehat{x}\|} \leq \kappa(A)\frac{\|E\|}{\|A\|}.$$

To estimate the error relative to $x$ is more involved and is similar to the estimate derived below.

One can show that if $\widehat{x}$ is the solution of $(A+E)\widehat{x} = \widehat{b}$ with both $A$ and $b$ perturbed, then

$$\frac{\|e\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \kappa(A)\|E\|/\|A\|}\left(\frac{\|E\|}{\|A\|} + \frac{\|r\|}{\|b\|}\right).$$

To establish this relationship use $(A+E)x = b + Ex$ and $(A+E)\widehat{x} = \widehat{b}$ to show $x - \widehat{x} = (A+E)^{-1}(Ex + r)$, and also use $\|r\| \leq \frac{\|r\|}{\|b\|}\|A\| \cdot \|x\|$. Note that if $\kappa(A)\frac{\|E\|}{\|A\|} = \|A^{-1}\| \cdot \|E\|$ is small, then $\frac{\kappa(A)}{1-\kappa(A)\|E\|/\|A\|} \approx \kappa(A)$.

We conclude this discussion by estimating the change in $A^{-1}$ due to a perturbation in $A$. Suppose $\|A^{-1}\| \cdot \|E\| < 1$. Then as above $A + E$ is invertible, and

$$\begin{aligned} A^{-1} - (A+E)^{-1} &= A^{-1} - \sum_{k=0}^{\infty} (-1)^k (A^{-1}E)^k A^1 \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} (A^{-1}E)^k A^{-1}, \end{aligned}$$

so

$$\begin{aligned} \|A^{-1} - (A+E)^{-1}\| &\leq \sum_{k=1}^{\infty} \|A^{-1}E\|^k \cdot \|A^{-1}\| \\ &= \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|} \|A^{-1}\| \\ &\leq \frac{\|A^{-1}\| \cdot \|E\|}{1 - \|A^{-1}\| \cdot \|E\|} \|A^{-1}\| \\ &= \frac{\kappa(A)}{1 - \kappa(A)\|E\|/\|A\|} \frac{\|E\|}{\|A\|} \|A^{-1}\|. \end{aligned}$$

So the relative error in the inverse satisfies

$$\frac{\|A^{-1} - (A+E)^{-1}\|}{\|A^{-1}\|} \leq \frac{\kappa(A)}{1 - \kappa(A)\|E\|/\|A\|} \frac{\|E\|}{\|A\|}.$$

Again, if $\kappa(A) \frac{\|E\|}{\|A\|}$ is small, then the relative error in the inverse is bounded (approximately) by the condition number $\kappa(A)$ of $A$ times the relative error $\frac{\|E\|}{\|A\|}$ in the matrix $A$.