

## Norms

A norm is a way of measuring the length of a vector. Let  $V$  be a vector space. A *norm* on  $V$  is a function  $\|\cdot\| : V \rightarrow [0, \infty)$  satisfying

- (i)  $(\forall v \in V) \|v\| \geq 0$ , and  $\|v\| = 0$  iff  $v = 0$
- (ii)  $(\forall \alpha \in \mathbb{F})(\forall v \in V) \|\alpha v\| = |\alpha| \cdot \|v\|$ , and
- (iii) (triangle inequality)  $(\forall v, w \in V) \|v + w\| \leq \|v\| + \|w\|$ .

The pair  $(V, \|\cdot\|)$  is called a *normed linear space* (or normed vector space).

**Fact.** A norm  $\|\cdot\|$  on a vector space  $V$  induces a metric  $d$  on  $V$  by

$$d(v, w) = \|v - w\|.$$

*Exercise.* Show  $d$  is a metric on  $V$ . All topological properties (e.g. open sets, closed sets,

convergence of sequences, continuity of functions, compactness, etc.) will refer to those of the metric space  $(V, d)$ .

*Examples.*

(1)  $\ell^p$  norm on  $\mathbb{F}^n$  ( $1 \leq p \leq \infty$ )

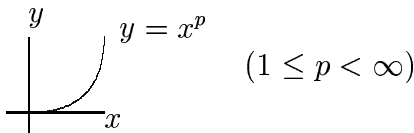
(a)  $p = \infty$ :  $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ ,  $x \in \mathbb{F}^n$

(b)  $1 \leq p < \infty$ :  $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ ,  $x \in \mathbb{F}^n$ .

The triangle inequality

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

is known as “Minkowski’s inequality.” It is a consequence of Hölder’s inequality. Integral versions of these inequalities are proved in real analysis texts, e.g., Folland or Royden. The proofs for vectors in  $\mathbb{F}^n$  are analogous to the proofs for integrals

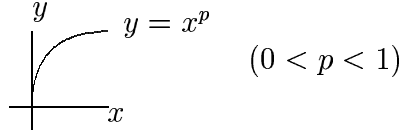


Related observation: for  $1 \leq p < \infty$ , the map  $x \mapsto x^p$  for  $x \geq 0$  is convex.

(c)  $0 < p < 1$ :  $(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$  is *not* a norm on  $\mathbb{F}^n$ . If  $x = e_1$ , and  $y = e_2$ ,

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} = 2^{\frac{1}{p}} > 2 = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}},$$

so the triangle inequality does not hold.



Related observation: for  $0 < p < 1$ , the map  $x \mapsto x^p$  for  $x \geq 0$  is *not* convex.

(2)  $\ell^p$  norm on  $\ell^p$  (subspace of  $\mathbb{F}^\infty$ ) ( $1 \leq p \leq \infty$ )

- (a)  $p = \infty$ :  $\ell^\infty = \{x \in \mathbb{F}^\infty : \sup_{i \geq 1} |x_i| < \infty\}$ ,  $\|x\|_\infty = \sup_{i \geq 1} |x_i|$  for  $x \in \ell^\infty$ .
- (b)  $1 \leq p < \infty$ :  $\ell^p = \left\{x \in \mathbb{F}^\infty : \left(\sum_{i=1}^\infty |x_i|^p\right)^{\frac{1}{p}} < \infty\right\}$ ,  $\|x\|_p = \left(\sum_{i=1}^\infty |x_i|^p\right)^{\frac{1}{p}}$  for  $x \in \ell^p$ . *Exercise.* Show that the triangle inequality follows from the finite-dimensional case.

(3)  $L^p$  norm on  $C([a, b])$  ( $1 \leq p \leq \infty$ )

- (a)  $p = \infty$ :  $\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$ .

Since  $|f(x)|$  is a continuous, real-valued function on the compact set  $[a, b]$ , it takes on its maximum, so the “sup” is actually a “max” here:

$$\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|.$$

- (b)  $1 \leq p < \infty$ :  $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$ .

Use continuity of  $f$  to show that  $\|f\|_p = 0 \Rightarrow f(x) \equiv 0$  on  $[a, b]$ . The triangle inequality

$$\left(\int_a^b |f(x) + g(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}$$

is Minkowski's inequality, a consequence of Hölder's inequality.

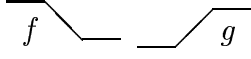
- (c)  $0 < p < 1$ :  $\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$  is *not* a norm on  $C([a, b])$ .

“Pseudo-example”: Let  $a = 0$ ,  $b = 1$ ,

$$f(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ 0 & \frac{1}{2} < x \leq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1. \end{cases}$$

Then

$$\begin{aligned} \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 |g(x)|^p dx\right)^{\frac{1}{p}} &= \left(\frac{1}{2}\right)^{\frac{1}{p}} + \left(\frac{1}{2}\right)^{\frac{1}{p}} \\ &= 2^{1-\frac{1}{p}} \\ &< 1 \\ &= \left(\int_0^1 |f(x) + g(x)|^p dx\right)^{\frac{1}{p}}, \end{aligned}$$

so the triangle inequality fails. Here,  $f$  and  $g$  are not continuous. *Exercise.* Adjust these  $f$  and  $g$  to be continuous (e.g.,  $\overline{f}$  ) to construct a legitimate counterexample to the triangle inequality.

*Remark.* There is also a Minkowski inequality for integrals: if  $1 \leq p < \infty$  and  $u \in C([a, b] \times [c, d])$ , then

$$\left( \int_a^b \left| \int_c^d u(x, y) dy \right|^p dx \right)^{\frac{1}{p}} \leq \int_c^d \left( \int_a^b |u(x, y)|^p dx \right)^{\frac{1}{p}} dy .$$

### Continuous Linear Operators on Normed Linear Spaces

**Theorem.** Suppose  $(V, \|\cdot\|_v)$  and  $W, \|\cdot\|_w)$  are normed linear spaces, and  $L : V \rightarrow W$  is a linear transformation. Then the following are equivalent:

- (a)  $L$  is continuous
- (b)  $L$  is uniformly continuous
- (c)  $(\exists C)$  so that  $(\forall v \in V) \|Lv\|_w \leq C\|v\|_v$ .

**Proof.** (a)  $\Rightarrow$  (c): Suppose  $L$  is continuous. Then  $L$  is continuous at  $v = 0$ . Let  $\epsilon = 1$ . Then  $\exists \delta > 0$  so that if  $\|v\|_v \leq \delta$ , then  $\|Lv\|_w \leq 1$  (as  $L(0) = 0$ ). For any  $v \neq 0$ ,  $\left\| \frac{\delta}{\|v\|_v} v \right\|_v \leq \delta$ , so  $\left\| L \left( \frac{\delta}{\|v\|_v} v \right) \right\|_w \leq 1$ , i.e.,  $\|Lv\|_w \leq \frac{1}{\delta} \|v\|_v$ . Let  $C = \frac{1}{\delta}$ .

(c)  $\Rightarrow$  (b): Suppose  $(\forall v \in V) \|Lv\|_w \leq C\|v\|_v$ . Then  $(\forall v_1, v_2 \in V) \|Lv_1 - Lv_2\|_w = \|L(v_1 - v_2)\|_w \leq C\|v_1 - v_2\|_v$ . Hence  $L$  is uniformly continuous (given  $\epsilon$ , let  $\delta = \frac{\epsilon}{C}$ , etc.). In fact,  $L$  is uniformly Lipschitz continuous with Lipschitz constant  $C$ .

(b) $\Rightarrow$ (a) is immediate.

**Definition.** If  $L : V \rightarrow W$  is a linear operator (where  $V$  and  $W$  are normed linear spaces), and  $\sup_{v \in V, v \neq 0} \frac{\|Lv\|_w}{\|v\|_v} < \infty$ , then  $L$  is called a *bounded linear operator* from  $V$  to  $W$ .

*Remarks.*

- (1) Note that it is the *norm ratio*  $\frac{\|Lv\|_w}{\|v\|_v}$  (or “stretching factor”) that is bounded, *not*  $\{\|Lv\|_w : v \in V\}$ .

*Exercise.* Show that if  $(\exists K) (\forall v \in V) \|Lv\|_w \leq K$ , then  $L \equiv 0$ .

- (2) The theorem above says that if  $V$  and  $W$  are normed linear spaces and  $L : V \rightarrow W$  is linear, then  $L$  is continuous  $\Leftrightarrow L$  is uniformly cont  $\Leftrightarrow L$  is a bounded linear operator.

**Definition.** If  $V$  and  $W$  are normed linear spaces and  $L : V \rightarrow W$  is a bounded linear operator, define the *operator norm* of  $L$  to be

$$\|L\| = \sup_{v \in V, v \neq 0} \frac{\|Lv\|_w}{\|v\|_v}.$$

*Remarks.*

- (1) There are other equivalent definitions of the operator norm  $\|L\|$ :

$$\begin{aligned} \|L\| &= \sup_{v \in V, \|v\|_v=1} \|Lv\|_w \\ \|L\| &= \sup_{v \in V, \|v\|_v \leq 1} \|Lv\|_w \\ \|L\| &= \min\{C : (\forall v \in V) \|Lv\|_w \leq C\|v\|_v\} \end{aligned}$$

(i.e.  $\|L\|$  is the smallest “stretching factor upper bound”  $C$ ).

*Exercise.* Show these are equivalent to the definition above.

- (2) The most common use of the operator norm is the obvious but powerful inequality:  
 $(\forall v \in V) \|Lv\|_w \leq \|L\| \cdot \|v\|_v.$

## Equivalence of Norms

**Lemma.** If  $(V, \|\cdot\|)$  is a normed linear space, then  $\|\cdot\| : (V, \|\cdot\|) \rightarrow \mathbb{R}$  is continuous.

**Proof.** For  $v_1, v_2 \in V$ ,  $\|v_1\| = \|v_1 - v_2 + v_2\| \leq \|v_1 - v_2\| + \|v_2\|$ , and thus  $\|v_1\| - \|v_2\| \leq \|v_1 - v_2\|$ . Similarly,  $\|v_2\| - \|v_1\| \leq \|v_2 - v_1\| = \|v_1 - v_2\|$ . So  $|\|v_1\| - \|v_2\|| \leq \|v_1 - v_2\|$ . Given  $\epsilon > 0$ , let  $\delta = \epsilon$ , etc.

**Definition.** Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ , both on the same vector space  $V$ , are called *equivalent norms* on  $V$  if  $\exists$  constants  $C_1, C_2 > 0$  for which  $(\forall v \in V) \frac{1}{C_1}\|v\|_2 \leq \|v\|_1 \leq C_2\|v\|_2$ .

*Remarks.*

- (1) Two norms  $\|\cdot\|_\alpha$  and  $\|\cdot\|_\beta$  on  $V$  are equivalent iff the identity map  $I : (V, \|\cdot\|_\alpha) \rightarrow (V, \|\cdot\|_\beta)$  is bicontinuous ( $\|v\|_\beta \leq C_1\|v\|_\alpha \Rightarrow I : (V, \|\cdot\|_\alpha) \rightarrow (V, \|\cdot\|_\beta)$  is continuous, and  $\|v\|_\alpha \leq C_2\|v\|_\beta \Rightarrow I : (V, \|\cdot\|_\beta) \rightarrow (V, \|\cdot\|_\alpha)$  is continuous.)
- (2) Equivalence of norms (denoted temporarily by  $\sim$ ) is an equivalence relation on the set of all norms on a fixed vector space  $V$ : (i)  $\|\cdot\|_\alpha \sim \|\cdot\|_\alpha$ ; (ii)  $\|\cdot\|_\alpha \sim \|\cdot\|_\beta$  iff  $\|\cdot\|_\beta \sim \|\cdot\|_\alpha$ ; and (iii) if  $\|\cdot\|_\alpha \sim \|\cdot\|_\beta$  and  $\|\cdot\|_\beta \sim \|\cdot\|_\gamma$ , then  $\|\cdot\|_\alpha \sim \|\cdot\|_\gamma$ .

For *finite dimensional* vector spaces  $V$ , all norms are equivalent.

## The Norm Equivalence Theorem

If  $V$  is a finite dimensional vector space, then any two norms on  $V$  are equivalent.

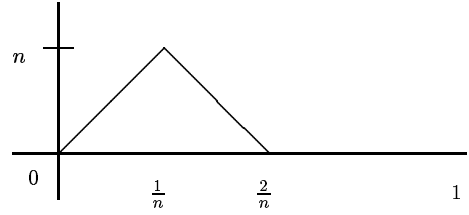
**Proof.** Fix a basis  $\{v_1, \dots, v_n\}$  for  $V$ , and identify  $V$  with  $\mathbb{F}^n$  ( $v \in V \leftrightarrow x \in \mathbb{F}^n$  where  $v = x_1v_1 + \dots + x_nv_n$ ). Using this identification, we can restrict our attention to  $\mathbb{F}^n$ . Let  $|x| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$  denote the euclidean norm [i.e.,  $\ell^2$  norm] on  $\mathbb{F}^n$ . Because equivalence of norms is an equivalence relation, it suffices to show that any given norm  $\|\cdot\|$  on  $\mathbb{F}^n$  is equivalent to the euclidean norm  $|\cdot|$ . For  $x \in \mathbb{F}^n$ ,  $\|x\| = \|\sum_{i=1}^n x_i e_i\| \leq \sum_{i=1}^n |x_i| \cdot \|e_i\| \leq (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}} (\sum_{i=1}^n \|e_i\|^2)^{\frac{1}{2}}$  by the Schwarz inequality in  $\mathbb{R}^n$ . Thus  $\|x\| \leq M|x|$ , where  $M = (\sum_{i=1}^n \|e_i\|^2)^{\frac{1}{2}}$ . Thus the identity map  $I : (\mathbb{F}^n, |\cdot|) \rightarrow (\mathbb{F}^n, \|\cdot\|)$  is continuous, which is half of what we have to show. Composing the map with  $\|\cdot\| : (\mathbb{F}^n, \|\cdot\|) \rightarrow \mathbb{R}$  (which is continuous by the preceding Lemma), we conclude that  $\|\cdot\| : (\mathbb{F}^n, |\cdot|) \rightarrow \mathbb{R}$  is continuous. Let  $S = \{x \in \mathbb{F}^n : |x| = 1\}$ . Then  $S$  is compact in  $(\mathbb{F}^n, |\cdot|)$ , and thus  $\|\cdot\|$  takes on its minimum on  $S$ , which must be  $> 0$  since  $0 \notin S$ . Let  $m = \min_{\{w:|w|=1\}} \|w\| > 0$ . Hence if  $|x| = 1$ , then  $\|x\| \geq m$ . For any  $x \in \mathbb{F}^n$  with  $x \neq 0$ ,  $|\frac{x}{|x|}| = 1$ , so  $\|\frac{x}{|x|}\| \geq m$ , i.e.  $|x| \leq \frac{1}{m}\|x\|$ . So  $\|\cdot\|$  and  $|\cdot|$  are equivalent.

*Remarks.*

- (1) All norms on a fixed finite dimensional vector space are equivalent. Be careful, though, when studying problems (e.g. in numerical PDE) where there is a sequence of finite dimensional spaces of increasing dimensions: the constants  $C_1$  and  $C_2$  in the equivalence can depend on the dimension (e.g.  $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$  in  $\mathbb{F}^n$ ).
- (2) The Norm Equivalence Theorem is *not* true in infinite dimensional vector spaces.
- (3) It can be shown that, for a normed linear space  $V$ , the closed unit ball  $\{v \in V : \|v\| \leq 1\}$  is compact iff  $\dim V < \infty$ .

*Examples.*

- (1) On  $\mathbb{F}_0^\infty = \{x \in \mathbb{F}^\infty : (\exists N)(\forall n \geq N) x_n = 0\}$ , for  $1 \leq p < q \leq \infty$ , the  $\ell^p$  norm and  $\ell^q$  norm are *not* equivalent. We show the case  $p = 1, q = \infty$ . First note that  $\|x\|_\infty \leq \sum_{i=1}^\infty |x_i| = \|x\|_1$ , so  $I : (\mathbb{F}_0^\infty, \|\cdot\|_1) \rightarrow (\mathbb{F}_0^\infty, \|\cdot\|_\infty)$  is continuous. But if  $y_1 = (1, 0, 0, \dots)$ ,  $y_2 = (1, 1, 0, \dots)$ ,  $y_3 = (1, 1, 1, 0, \dots)$ , etc., then  $\|y_n\|_\infty = 1 \forall n$ , but  $\|y_n\|_1 = n$ . So there does *not* exist a constant  $C$  for which  $(\forall x \in \mathbb{F}_0^\infty) \|x\|_1 \leq C\|x\|_\infty$ .
- (2) On  $C([a, b])$ , for  $1 \leq p < q \leq \infty$ , the  $L^p$  and  $L^q$  norms are *not* equivalent. We will show the case  $p = 1, q = \infty$  here:  $\|u\|_1 = \int_a^b |u(x)|dx \leq \int_a^b \|u\|_\infty dx = (b-a)\|u\|_\infty$ , so  $I : (C([a, b]), \|\cdot\|_\infty) \rightarrow (C([a, b]), \|\cdot\|_1)$  is continuous. (Remark: Since the integral  $\mathcal{I}(u) = \int_a^b u(x)dx$  is clearly continuous on  $(C([a, b]), \|\cdot\|_1)$  since  $|\mathcal{I}(u_1) - \mathcal{I}(u_2)| \leq \int_a^b |u_1(x) - u_2(x)|dx = \|u_1 - u_2\|_1$ , composition of these two continuous operators implies the standard result that if  $u_n \rightarrow u$  uniformly on  $[a, b]$ , then  $\int_a^b u_n(x)dx \rightarrow \int_a^b u(x)dx$ .) WLOG assume  $a = 0, b = 1$ . Let  $u_n$  be



Then  $\|u_n\|_1 = 1$ , but  $\|u_n\|_\infty = n$ . So there does not exist a constant  $C$  for which  $(\forall u \in C([a, b])) \|u\|_\infty \leq C\|u\|_1$ .

- (3) In  $\ell^2$  (subspace of  $\mathbb{F}^\infty$ ) with norm  $\|x\|_{\ell^2} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ , the closed unit ball  $\{x \in \ell^2 : \|x\|_{\ell^2} \leq 1\}$  is *not* compact. The sequence  $e_1, e_2, e_3, \dots$  is bounded  $\|e_i\| \leq 1$ , and all are in the closed unit ball, but no subsequence converges because  $\|e_i - e_j\|_{\ell^2} = \sqrt{2}$  for  $i \neq j$ .

*Exercise.* Does the sequence  $e_1, e_2, e_3, \dots$  converge weakly in  $\ell^2$ ? (A sequence  $\{x_n\}$  is said to converge weakly to  $x$  in  $\ell^2$  if  $(\forall y \in \ell^2) \langle x_n, y \rangle \rightarrow \langle x, y \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the inner product  $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$  on  $\ell^2$ .)

## Norms induced by inner products

Let  $V$  be a vector space and  $\langle \cdot, \cdot \rangle$  be an inner product on  $V$ . Define  $\|v\| = \sqrt{\langle v, v \rangle}$ . By the properties of an inner product,  $\|v\| \geq 0$  with  $\|v\| = 0$  iff  $v = 0$ , and  $(\forall \alpha \in \mathbb{F})(\forall v \in V) \|\alpha v\| = |\alpha| \cdot \|v\|$ . To show that  $\|\cdot\|$  is actually a norm on  $V$  we need the triangle inequality. We begin by first showing the *Cauchy-Schwarz inequality*.

**Lemma.**[The Cauchy-Schwarz inequality] For all  $v, w \in V$  we have  $|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$ . Moreover, we have equality iff  $v$  and  $w$  are linearly dependent. (This latter statement is sometimes called the “converse of Cauchy-Schwarz.”)

### Proof.

Case (i) If  $v = 0$  or  $w = 0$ , clear.

Case (ii) If  $\|v\| = \|w\| = 1$  and  $\langle v, w \rangle \geq 0$ , then  $0 \leq \|v - w\|^2 = \langle v - w, v - w \rangle = \langle v, v \rangle - 2\operatorname{Re}\langle v, w \rangle + \langle w, w \rangle = 2(1 - \langle v, w \rangle)$  so  $\langle v, w \rangle \leq 1$  (with equality iff  $v = w$ ).

Case (iii) For any  $v \neq 0$  and  $w \neq 0$ , choose  $\alpha \in \mathbb{F}$  with  $|\alpha| = 1$  and  $\alpha \langle v, w \rangle \geq 0$ . Let  $v_1 = \frac{\alpha}{\|v\|}v$  and  $w_1 = \frac{w}{\|w\|}$ . Then  $\|v_1\| = \|w_1\| = 1$  and  $\langle v_1, w_1 \rangle \geq 0$ , so  $\frac{|\langle v, w \rangle|}{\|v\| \cdot \|w\|} = \frac{\alpha \langle v, w \rangle}{\|v\| \cdot \|w\|} = \langle v_1, w_1 \rangle \leq 1$  (with equality iff  $v_1 = w_1$ ).

*Exercise.* In case (iii) of the above proof, show  $v, w$  are linearly dependent iff  $v_1 = w_1$ .

Now the triangle inequality follows

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle = \langle v, v \rangle + 2\operatorname{Re}\langle v, w \rangle + \langle w, w \rangle \\ &\leq \|v\|^2 + 2|\langle v, w \rangle| + \|w\|^2 \leq \|v\|^2 + 2\|v\| \cdot \|w\| + \|w\|^2 = (\|v\| + \|w\|)^2. \end{aligned}$$

So  $\|v\| = \sqrt{\langle v, v \rangle}$  is a norm on  $V$ . It is called the norm induced by the inner product  $\langle \cdot, \cdot \rangle$ . An inner product induces a norm, which induces a metric  $(V, \langle \cdot, \cdot \rangle) \leftrightarrow (V, \|\cdot\|) \leftrightarrow (V, d)$ .

*Examples.*

- (1) The Euclidean norm [i.e.  $\ell^2$  norm] on  $\mathbb{F}^n$  is induced by the standard inner product  $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i$ :  $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i \bar{x}_i} = \sqrt{\sum_{i=1}^n |x_i|^2}$ .
- (2) Let  $A \in \mathbb{F}^{n \times n}$  be Hermitian symmetric and positive definite, and let

$$\langle x, y \rangle_A = \sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} \bar{y}_j \quad \text{for } x, y \in \mathbb{F}^n.$$

Then  $\langle \cdot, \cdot \rangle_A$  is an inner product on  $\mathbb{F}^n$ , which induces the norm

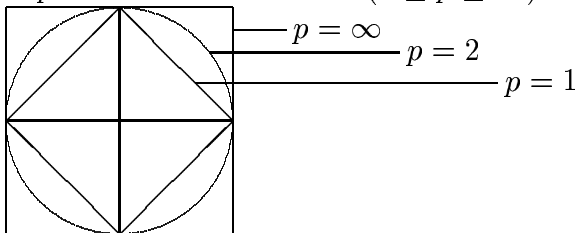
$$\|x\|_A = \sqrt{\langle x, x \rangle_A} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} \bar{x}_j} = \sqrt{x^T A \bar{x}} = \sqrt{x^H A x}.$$

*Remark.* An alternate convention is to define  $\langle x, y \rangle_A$  to be  $\sum_{i=1}^n \sum_{j=1}^n \bar{y}_i a_{ij} x_j = y^H A x$ , in which case  $\|x\|_A = \sqrt{x^H A x}$ .

- (3) The  $\ell^2$  norm on  $\ell^2$  (subspace of  $\mathbb{F}^\infty$ ) is induced by the inner product  $\langle x, y \rangle = \sum_{i=1}^\infty x_i \bar{y}_i$ :  $\|x\|_2 = \sqrt{\sum_{i=1}^\infty x_i \bar{x}_i} = \sqrt{\sum_{i=1}^\infty |x_i|^2}$ .
- (4) The  $L^2$  norm  $\|u\|_2 = \left( \int_a^b |u(x)|^2 dx \right)^{\frac{1}{2}}$  on  $C([a, b])$  is induced by the inner product  $\langle u, v \rangle = \int_a^b u(x) \bar{v}(x) dx$ .

### Closed unit balls $\{v \in V : \|v\| \leq 1\}$ in finite dimensional normed linear spaces $V$

*Example.* For  $\ell^p$  norms in  $\mathbb{R}^2$  ( $1 \leq p \leq \infty$ )

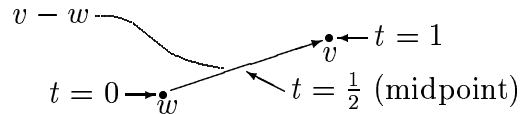


**Definition.** A subset  $C$  of a vector space  $V$  is called *convex* if

$$(\forall v, w \in C)(\forall t \in [0, 1]) \quad tv + (1 - t)u \in C.$$

*Remarks.*

- (1) This means that the line segment joining  $v$  and  $w$  is in  $C$  if  $v$  and  $w$  are in  $C$ :



$w + t(v - w)$  is on this line segment.

- (2) The linear combination  $tv + (1 - t)w$  for  $t \in [0, 1]$  is often called a *convex combination* of  $v$  and  $w$ .

Let  $B = \{v \in V : \|v\| \leq 1\}$  denote the closed unit ball in a *finite dimensional* normed linear space.

**Facts.**

- (1)  $B$  is convex.
- (2)  $B$  is compact.
- (3)  $B$  is symmetric (if  $v \in B$  and  $\alpha \in \mathbb{F}$  with  $|\alpha| = 1$ , then  $\alpha v \in B$ ).
- (4) The origin is in the interior of  $B$ .

**Lemma.** If  $\dim V < \infty$  and  $B \subset V$  satisfies the four conditions in the statement of *facts* above, then there is a unique norm on  $V$  for which  $B$  is the closed unit ball:

$$\|v\| = \inf\{c > 0 : \frac{v}{c} \in B\}.$$

*Remark.* The condition that 0 be in the interior of a set is independent of the norm: by the norm equivalence theorem, all norms induce the same topology on  $V$ , i.e. have the same collection of open sets.

*Exercise.* Show that the object defined in the lemma above does indeed define a norm, and that  $B$  is its closed unit ball. The uniqueness of this norm follows from the fact that in any normed linear space,  $\|v\| = \inf\{c > 0 : \frac{v}{c} \in B\}$  where  $B$  is the closed unit ball  $B = \{v : \|v\| \leq 1\}$ . Hence there is a one-to-one correspondence between norms on a finite dimensional vector space and subsets  $B$  satisfying the four conditions stated above.



## Completeness

Completeness in a normed linear space  $(V, \|\cdot\|)$  means completeness in the metric space  $(V, d)$ , where  $d(v, w) = \|v - w\|$ : every Cauchy sequence  $\{v_n\}$  in  $V$  (i.e.  $(\forall \epsilon > 0)(\exists N)(\forall n, m \geq N) \|v_n - v_m\| < \epsilon$ ) has a limit in  $V$  (i.e.  $(\exists v \in V)\|v_n - v\| \rightarrow 0$  as  $n \rightarrow \infty$ ).

*Example.*  $\mathbb{F}^n$  endowed with the euclidean norm  $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$  is complete.

Topological properties are those which depend only on the collection of open sets (e.g., open, closed, compact, whether a sequence converges, etc.). Completeness is *not* a topological property.

*Example.* Let  $f : [1, \infty) \rightarrow (0, 1]$  be given by  $f(x) = \frac{1}{x}$  (with the usual metric on  $\mathbb{R}$ ). Then  $f$  is a homeomorphism (bijective, bicontinuous), but  $[1, \infty)$  is complete while  $(0, 1]$  is *not* complete.

Completeness is a *uniform property*.

**Theorem.** If  $(X, \rho)$  and  $(Y, \sigma)$  are metric spaces, and  $\varphi : (X, \rho) \rightarrow (Y, \sigma)$  is a uniform homeomorphism (i.e., bijective, bicontinuous and  $\varphi$  and  $\varphi^{-1}$  are both uniformly continuous), then  $(X, \rho)$  is complete iff  $(Y, \sigma)$  is complete.

The key step in the proof of this theorem is to show that if  $\varphi : X \rightarrow Y$  is a uniform homeomorphism, then  $\varphi$  preserves Cauchy sequences, i.e. a sequence  $\{x_n\}$  is Cauchy in  $(X, \rho)$  iff  $\{\varphi(x_n)\}$  is Cauchy in  $(Y, \sigma)$ . Since bounded linear operators between normed linear spaces are automatically uniformly continuous, several facts follow immediately.

**Corollary.** If two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on a vector space  $V$  are equivalent, then  $(V, \|\cdot\|_1)$  is complete iff  $(V, \|\cdot\|_2)$  is complete.

**Corollary.** Every finite dimensional normed linear space is complete.

**Proof.** If  $\dim V = n < \infty$ , choose a basis of  $V$  and use it to identify  $V$  with  $\mathbb{F}^n$ . Since  $\mathbb{F}^n$  is complete in the euclidean norm, the corollary follows from the norm equivalence theorem.

But not every infinite dimensional normed linear space is complete.

**Definition.** A complete normed linear space is called a *Banach space*. An inner product space for which the induced norm is complete is called a *Hilbert space*.

*Examples.* To show that a normed linear space is complete, we must show that every Cauchy sequence converges in that space. The basic strategy for showing that a space is complete is a three step process that can be described as follows: given a Cauchy sequence,

- (i) construct what you think is its limit;
- (ii) show the limit is in the space  $V$ ;
- (iii) show the sequence converges to the limit in  $V$ .

- (1) Let  $M$  be a metric space. Let  $C(M)$  denote the vector space of continuous functions  $u : M \rightarrow \mathbb{F}$ . Let  $C_b(M)$  denote the subspace of  $C(M)$  consisting of all bounded continuous functions  $C_b(M) = \{u \in C(M) : (\exists K)(\forall x \in M)|u(x)| \leq K\}$ . On  $C_b(M)$ , define the sup-norm  $\|u\| = \sup_{x \in M} |u(x)|$ .

*Fact.*  $(C_b M, \|\cdot\|)$  is complete.

*Proof.* Let  $\{u_n\} \subset C_b(M)$  be Cauchy in  $\|\cdot\|$ . Given  $\epsilon > 0$ ,  $\exists N$  so that  $(\forall n, m \geq N) \|u_n - u_m\| < \epsilon$ . For each  $x \in M$ ,  $|u_n(x) - u_m(x)| \leq \|u_n - u_m\|$ , so for each  $x \in M$ ,  $\{u_n(x)\}$  is a Cauchy sequence in  $\mathbb{F}$ , which has a limit in  $\mathbb{F}$  (which we will call  $u(x)$ ) since  $\mathbb{F}$  is complete:  $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ . Given  $\epsilon > 0$ ,  $(\exists N)(\forall n, m \geq N)(\forall x \in M) |u_n(x) - u_m(x)| < \epsilon$ . Taking the limit (for each fixed  $x$ ) as  $m \rightarrow \infty$ , we get  $(\forall n \geq N)(\forall x \in M) |u_n(x) - u(x)| \leq \epsilon$ . Thus  $u_n \rightarrow u$  uniformly, so  $u$  is continuous (since the uniform limit of continuous functions is continuous). Clearly  $u$  is bounded (choose  $N$  for  $\epsilon = 1$ ; then  $(\forall x \in M) |u(x)| \leq \|u_N\| + 1$ ), so  $u \in C_b(M)$ . And now we have  $\|u_n - u\| \rightarrow 0$  as  $n \rightarrow \infty$ , i.e.,  $u_n \rightarrow u$  in  $(C_b(M), \|\cdot\|)$ .  $\square$

(2)  $\ell^p$  is complete for  $1 \leq p \leq \infty$ .

$p = \infty$ . This is a special case of (1) where  $M = \mathbb{N} = \{1, 2, 3, \dots\}$ .

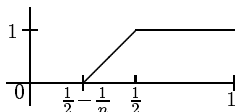
$1 \leq p < \infty$ . Let  $\{x_k\}$  be a Cauchy sequence in  $\ell^p$ ; write  $x_k = (x_{k_1}, x_{k_2}, \dots)$ . Given  $\epsilon > 0$ ,  $(\exists K)(\forall k, \ell \geq K) \|x_k - x_\ell\|_p < \epsilon$ . For each  $m \in \mathbb{N}$ ,

$$|x_{k_m} - x_{\ell_m}| \leq \left( \sum_{i=1}^{\infty} |x_{k_i} - x_{\ell_i}|^p \right)^{\frac{1}{p}} = \|x_k - x_\ell\|,$$

so for each  $m \in \mathbb{N}$ ,  $\{x_{k_m}\}_{k=1}^{\infty}$  is a Cauchy sequence in  $\mathbb{F}$ , which has a limit: let  $x_m = \lim_{k \rightarrow \infty} x_{k_m}$ . Let  $x$  be the sequence  $x = (x_1, x_2, x_3, \dots)$ ; so far, we just know that  $x \in \mathbb{F}^\infty$ . Given  $\epsilon > 0$ ,  $(\exists K)(\forall k, \ell \geq K) \|x_k - x_\ell\| < \epsilon$ . Then for any  $N$  and for  $k, \ell \geq K$ ,  $\left( \sum_{k=1}^N |x_{k_i} - x_{\ell_i}|^p \right)^{\frac{1}{p}} < \epsilon$ ; taking the limit as  $\ell \rightarrow \infty$ ,  $\left( \sum_{i=1}^N |x_{k_i} - x_i|^p \right)^{\frac{1}{p}} \leq \epsilon$ ; then taking the limit as  $N \rightarrow \infty$ ,  $\left( \sum_{i=1}^{\infty} |x_{k_i} - x_i|^p \right)^{\frac{1}{p}} \leq \epsilon$ . Thus  $x_K - x \in \ell^p$ , so also  $x = x_K - (x_K - x) \in \ell^p$ , and we have  $(\forall k \geq K) \|x_k - x\|_p \leq \epsilon$ . Thus  $\|x_k - x\|_p \rightarrow 0$  as  $k \rightarrow \infty$ , i.e.,  $x_k \rightarrow x$  in  $\ell^p$ .  $\square$

(3) If  $M$  is a compact metric space, then every continuous function  $u : M \rightarrow \mathbb{F}$  is bounded, so  $C(M) = C_b(M)$ . In particular,  $C(M)$  is complete in the sup norm  $\|u\| = \sup_{x \in M} |u(x)|$  (special case of (1).) For example,  $C([a, b])$  is complete in the  $L^\infty$  norm.

(4) For  $1 \leq p < \infty$ ,  $C([a, b])$  is *not* complete in the  $L^p$  norm.

*Example.* On  $[0, 1]$ , let  $u_n$  be:  Then  $u_n \in C[0, 1]$ .

*Exercise:* Show that  $\{u_n\}$  is Cauchy in  $\|\cdot\|_p$ . We must show that there does *not* exist a  $u \in C[0, 1]$  for which  $\|u_n - u\|_p \rightarrow 0$ .

*Exercise:* Show that if  $u \in C[0, 1]$  and  $\|u_n - u\|_p \rightarrow 0$ , then  $u(x) \equiv 0$  for  $0 \leq x < \frac{1}{2}$  and  $u(x) \equiv 1$  for  $\frac{1}{2} < x \leq 1$ , contradicting the continuity of  $u$  at  $x = \frac{1}{2}$ .

- (5)  $\mathbb{F}_0^\infty = \{x \in \mathbb{F}^\infty : (\exists N)(\forall n \geq N) x_n = 0\}$  is *not* complete in any  $\ell^p$  norm ( $1 \leq p \leq \infty$ ). This can be shown using the sequences described below.

$1 \leq p < \infty$ . Choose any  $x \in \ell^p \setminus \mathbb{F}_0^\infty$ , and consider the truncated sequences  $y_1 = (x_1, 0, \dots)$ ;  $y_2 = (x_1, x_2, 0, \dots)$ ;  $y_3 = (x_1, x_2, x_3, 0, \dots)$ ; etc.

*Exercise:* Show that  $\{y_n\}$  is Cauchy in  $(\mathbb{F}_0^\infty, \|\cdot\|_p)$ , but that there is no  $y \in \mathbb{F}_0^\infty$  for which  $\|y_n - y\|_p \rightarrow 0$ .

$p = \infty$ . Same idea: choose any  $x \in \ell^\infty \setminus \mathbb{F}_0^\infty$  for which  $\lim_{i \rightarrow \infty} x_i = 0$ , and consider the sequence of truncated sequences.

## Completion of a Metric Space

**Fact.** Let  $(X, \rho)$  be a metric space. Then there exists a complete metric space  $(\bar{X}, \bar{\rho})$  and an “inclusion map”  $i : X \rightarrow \bar{X}$  for which  $i$  is injective,  $i$  is an isometry from  $X$  to  $i[X]$  (i.e.  $(\forall x, y \in X) \rho(x, y) = \bar{\rho}(i(x), i(y))$ ), and  $i[X]$  is dense in  $\bar{X}$ . Moreover, all such  $(\bar{X}, \bar{\rho})$  are isometrically isomorphic. The metric space  $(\bar{X}, \bar{\rho})$  is called the *completion* of  $(X, \rho)$ .

One way to construct such an  $\bar{X}$  is to take equivalence classes of Cauchy sequences in  $X$  to be elements of  $\bar{X}$ .

### Representations of Completions

In some situations, the completion of a metric space can be identified with a larger vector space which actually includes  $X$ , and whose elements are objects of a similar nature to the elements of  $X$ . One example is  $\mathbb{R} =$  completion of the rationals  $\mathbb{Q}$ . The completion of  $C([a, b])$  in the  $L^p$  norm (for  $1 \leq p < \infty$ ) can be represented as  $L^p([a, b])$ , the vector space of [equivalence classes of] Lebesgue measurable functions  $u : [a, b] \rightarrow \mathbb{F}$  for which  $\int_a^b |u(x)|^p dx < \infty$ , with norm  $\|u\|_p = \left(\int_a^b |u(x)|^p dx\right)^{\frac{1}{p}}$ .

**Fact.** A subset of a complete metric space is complete iff it is closed.

**Proposition.** Let  $V$  be a Banach space, and  $W \subset V$  be a subspace. The norm on  $V$  restricts to a norm on  $W$ . We have:

$$W \text{ is complete} \quad \text{iff} \quad W \text{ is closed.}$$

*Examples.*

- (1)  $C_0(\mathbb{R}^n) = \{u \in C_b(\mathbb{R}^n) : \lim_{|x| \rightarrow \infty} u(x) = 0\}$ .
- (2)  $C_c(\mathbb{R}^n) = \{u \in C_b(\mathbb{R}^n) : (\exists K > 0) \ni (\forall x \text{ with } |x| \geq K) u(x) = 0\}$ .

*Remarks.*

- (1) If  $M$  is a metric space and  $u : M \rightarrow \mathbb{F}$  is a function, define the *support* of  $u$  to be the closure of  $\{x \in M : u(x) \neq 0\}$ . The support of a function is automatically closed. The complement of the support of a function is the interior of  $\{x \in M : u(x) = 0\}$ .
- (2) Elements of  $C_c(\mathbb{R}^n)$  are continuous functions with *compact support*.
- (3)  $C_0(\mathbb{R}^n)$  is complete in the sup norm (exercise). This can either be shown directly, or by showing that  $C_0(\mathbb{R}^n)$  is a closed subspace of  $C_b(\mathbb{R}^n)$ .
- (4)  $C_c(\mathbb{R}^n)$  is *not* complete. In fact,  $C_c(\mathbb{R}^n)$  is dense in  $C_0(\mathbb{R}^n)$ . So  $C_0(\mathbb{R}^n)$  is a representation of the completion of  $C_c(\mathbb{R}^n)$  in the sup norm.

## Series in normed linear spaces

Let  $(V, \|\cdot\|)$  be a normed linear space. Consider a series  $\sum_{n=1}^{\infty} v_n$  in  $V$ .

**Definition.** We say the series *converges in  $V$*  if  $\exists v \in V \ni \lim_{N \rightarrow \infty} \|S_N - v\| = 0$ , where  $S_N = \sum_{n=1}^N v_n$  is the  $N^{\text{th}}$  partial sum. We say this series *converges absolutely* if  $\sum_{n=1}^{\infty} \|v_n\| < \infty$ .

*Caution:* Strictly speaking, if a series “converges absolutely” in a normed linear space, it does not have to converge in that space.

*Example.* The series  $(1, 0 \cdots) + (0, \frac{1}{2}, 0 \cdots) + (0, 0, \frac{1}{4}, 0 \cdots)$  “converges absolutely” in  $\mathbb{F}_0^\infty$ , but it doesn’t converge in  $\mathbb{F}_0^\infty$ .

**Proposition.** A normed linear space  $(V, \|\cdot\|)$  is complete iff every absolutely convergent series actually converges in  $(V, \|\cdot\|)$ .

*Proof Sketch* ( $\Rightarrow$ ) Given an absolutely convergent series, show that the sequence of partial sums is Cauchy: for  $m > n$   $\|S_m - S_n\| \leq \sum_{j=n+1}^m \|v_j\|$ .

( $\Leftarrow$ ) Given a Cauchy sequence  $\{x_n\}$ , choose  $n_1, n_2 < \cdots$  inductively so that for  $k = 1, 2, \dots$ ,  $(\forall n, m \geq n_k) \|x_n - x_m\| \leq 2^{-k}$ . Then in particular  $\|x_{n_k} - x_{n_{k+1}}\| \leq 2^{-k}$ . Show that the series  $x_{n_1} + \sum_{k=2}^{\infty} (x_{n_k} - x_{n_{k-1}})$  is absolutely convergent. Let  $x$  be its limit. Show that  $x_n \rightarrow x$ .