Completeness

\((V, \| \cdot \|)\) a normed linear space.

We say \((V, \| \cdot \|)\) is complete if every Cauchy sequence in \(V\) has a limit in \(V\).

\(\{v_n\} \subset V\) is Cauchy if

\[(\forall \epsilon > 0)(\exists N)(\forall n, m \geq N) \quad \|v_n - v_m\| < \epsilon.\]

\(\{v_n\} \subset V\) has limit \(v\) if \(\|v - v_n\| \to 0\).

For example, \(\mathbb{F}^n\) endowed with the Euclidean norm

\[\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}\]

is complete.

Topological properties are those that depend only on the collection of open sets (e.g., open, closed, compact, whether a sequence converges, etc.). Completeness is not a topological property.

Example:
Let \(f: [1, \infty) \to (0, 1]\) be given by \(f(x) = \frac{1}{x}\) (with the usual metric on \(\mathbb{R}\)). Then \(f\) is a homeomorphism (bijective, bicontinuous), but \([1, \infty)\) is complete while \((0, 1]\) is not complete.

Completeness is a uniform property.

**Theorem.** If \((X, \rho)\) and \((Y, \sigma)\) are metric spaces, and

\[\varphi: (X, \rho) \to (Y, \sigma)\]

is a uniform homeomorphism (i.e., bijective, bicontinuous and \(\varphi\) and \(\varphi^{-1}\) are both uniformly continuous), then \((X, \rho)\) is complete iff \((Y, \sigma)\) is complete.
Since bounded linear operators between normed linear spaces are automatically uniformly continuous, several facts follow immediately.

**Corollary.** If two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a vector space $V$ are equivalent, then $(V, \| \cdot \|_1)$ is complete iff $(V, \| \cdot \|_2)$ is complete.

**Corollary.** Every finite-dim normed linear space is complete.

But not every infinite-dim normed linear space is complete.

**Definition.** A complete normed linear space is called a *Banach space*. An inner product space for which the induced norm is complete is called a *Hilbert space*.

To show that a normed linear space is complete, we must show that every Cauchy sequence converges in that space.

The basic strategy for showing completeness is a three step process that can be described as follows:

Given a Cauchy sequence,

(i) construct what you think is its limit;

(ii) show the limit is in the space $V$;

(iii) show the sequence converges to the limit in $V$. 
Examples

- \((C_b(M), \| \cdot \|)\)

\(M\) a metric space.
\(C(M)\) the vector space of continuous functions \(u : M \to \mathbb{F}\).
\(C_b(M)\) the subspace of \(C(M)\) of all bounded continuous functions
\[C_b(M) = \{ u \in C(M) : (\exists K)(\forall x \in M) \ |u(x)| \leq K \} .\]
On \(C_b(M)\), define the sup-norm
\[\|u\| = \sup_{x \in M} |u(x)|.\]

Fact \((C_b(M), \| \cdot \|)\) is complete.

Proof Let \(\{u_n\} \subset C_b(M)\) be Cauchy in \(\| \cdot \|\).
Given \(\varepsilon > 0\), \(\exists N\) so that
\[(\forall n, m \geq N) \ |u_n - u_m| < \varepsilon.\]
For each \(x \in M\), \(|u_n(x) - u_m(x)| \leq \|u_n - u_m\|\), so for each \(x \in M\),
\(\{u_n(x)\}\) is a Cauchy sequence in \(\mathbb{F}\), which has a limit in \(\mathbb{F}\) (which we
will call \(u(x)\)) since \(\mathbb{F}\) is complete:
\[u(x) = \lim_{n \to \infty} u_n(x).\]
Let \(\varepsilon > 0\), then
\[(\exists N)(\forall n, m \geq N)(\forall x \in M) \ |u_n(x) - u_m(x)| < \varepsilon.\]
Take the limit (for each fixed \(x\)) to get
\[(\forall n \geq N)(\forall x \in M) \ |u_n(x) - u(x)| \leq \varepsilon.\]
Thus \(u_n \to u\) uniformly, so \(u\) is continuous (since the uniform limit
of continuous functions is continuous). Clearly \(u\) is bounded, so
\(u \in C_b(M)\). And now we have \(\|u_n - u\| \to 0\) as \(n \to \infty\), i.e.,
\(u_n \to u\) in \((C_b(M), \| \cdot \|)\). \(\square\)
• \( \ell^p \) is complete for \( 1 \leq p \leq \infty \).

\( p = \infty \). This is a special case of (1) where \( M = \mathbb{N} = \{1, 2, 3, \ldots \} \).

\( 1 \leq p < \infty \). Let \( \{x_k\} \subset \ell^p \) be Cauchy.

Write \( x_k = (x_{k1}, x_{k2}, \ldots) \). Given \( \epsilon > 0 \),

\[
(\exists K)(\forall k, \ell \geq K) \quad \|x_k - x_\ell\|_p < \epsilon.
\]

For each \( m \in \mathbb{N} \),

\[
|x_{km} - x_{\ell m}| \leq \left( \sum_{i=1}^{\infty} |x_{ki} - x_{\ell i}|^p \right)^{1/p} = \|x_k - x_\ell\|,
\]

so for each \( m \in \mathbb{N} \), \( \{x_{km}\}_{k=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{F} \), which has a limit: let \( x_m = \lim_{k \to \infty} x_{km} \).

Let \( x \) be the sequence \( x = (x_1, x_2, x_3, \ldots) \); so far, we just know that \( x \in \ell^\infty \). Given \( \epsilon > 0 \),

\[
(\exists K)(\forall k, \ell \geq K) \quad \|x_k - x_\ell\| < \epsilon.
\]

Then for any \( N \) and for \( k, \ell \geq K \),

\[
\left( \sum_{i=1}^{N} |x_{ki} - x_{\ell i}|^p \right)^{1/p} < \epsilon.
\]

Taking the limit in \( \ell \),

\[
\left( \sum_{i=1}^{N} |x_{ki} - x_i|^p \right)^{1/p} \leq \epsilon.
\]

Taking the limit in \( N \),

\[
\left( \sum_{i=1}^{\infty} |x_{ki} - x_i|^p \right)^{1/p} \leq \epsilon.
\]

Thus \( x_k - x \in \ell^p \), so also \( x = x_k - (x_k - x) \in \ell^p \), and we have

\[
(\forall k \geq K) \quad \|x_k - x\|_p \leq \epsilon.
\]

Thus \( \|x_k - x\|_p \to 0 \) as \( k \to \infty \), i.e., \( x_k \to x \) in \( \ell^p \). \( \square \)
\[ F_0^\infty = \{ x \in F^\infty : (\exists N)(\forall n \geq N) \quad x_n = 0 \} \]

is not complete in any \( \ell^p \) norm \( (1 \leq p \leq \infty) \).

\[ 1 \leq p < \infty. \]
Choose any \( x \in \ell^p \setminus F_0^\infty \), and consider the truncated sequences

\[ y_1 = (x_1, 0, \ldots), \quad y_2 = (x_1, x_2, 0, \ldots), \quad y_3 = (x_1, x_2, x_3, 0, \ldots), \ldots \]

\( \{y_n\} \) is Cauchy in \( (F_0^\infty, \| \cdot \|_p) \), but there is no \( y \in F_0^\infty \) for which \( \|y_n - y\|_p \to 0 \).

\[ p = \infty. \]
Same idea: choose any \( x \in \ell^\infty \setminus F_0^\infty \) for which

\[ \lim_{i \to \infty} x_i = 0, \]

and consider the sequence of truncated sequences.
Every Metric Space can be Completed

**Fact.** Let \((X, \rho)\) be a metric space. Then there exists a complete metric space \((\bar{X}, \bar{\rho})\) and an “inclusion map” \(i : X \to \bar{X}\) for which

\[
i \text{ is injective},
\]

\[
i \text{ is an isometry from } X \text{ to } i[X],
\[
(\forall x, y \in X) \rho(x, y) = \bar{\rho}(i(x), i(y)),
\]

and

\[
i[X] \text{ is dense in } \bar{X}.
\]

Moreover, all such \((\bar{X}, \bar{\rho})\) are isometrically isomorphic. The metric space \((\bar{X}, \bar{\rho})\) is called the *completion* of \((X, \rho)\).

One way to construct such an \(\bar{X}\) is to take equivalence classes of Cauchy sequences in \(X\) to be elements of \(\bar{X}\).
Representations of Completions

Sometimes the completion of a metric space can be identified with a larger vector space which actually includes $X$, and whose elements are objects of a similar nature to the elements of $X$. One example is $\mathbb{R} = \text{completion of the rationals } \mathbb{Q}$.

The completion of $C([a, b])$ in the $L^p$ norm (for $1 \leq p < \infty$) is denoted by $L^p([a, b])$.

$L^p([a, b])$ is the vector space of equivalence classes of Lebesgue measurable functions $u : [a, b] \rightarrow \mathbb{F}$ for which $\int_a^b |u(x)|^p dx < \infty$, with norm

$$||u||_p = \left( \int_a^b |u(x)|^p dx \right)^{\frac{1}{p}}.$$

**Fact.** A subset of a complete metric space is complete iff it is closed.

**Proposition.** Let $V$ be a Banach space, and $W \subset V$ be a subspace. The norm on $V$ restricts to a norm on $W$. We have:

$W$ is complete iff $W$ is closed.
Examples of Complete Spaces as Closed Subspaces

Consider the spaces

\[ C_0(\mathbb{R}^n) \quad \text{and} \quad C_c(\mathbb{R}^n). \]

\[ C_0(\mathbb{R}^n) = \{ u \in C_b(\mathbb{R}^n) : \lim_{|x| \to \infty} u(x) = 0 \} \]

\[ C_c(\mathbb{R}^n) = \{ u \in C^*_b(\mathbb{R}^n) : (\exists K > 0) \exists (\forall x \text{ with } |x| \geq K) u(x) = 0 \} \]

- Suppose \( M \) is a metric space and \( u : M \to \mathbb{F} \) is a function. The support of \( u \) is the closure of \( \{ x \in M : u(x) \neq 0 \} \). The complement of the support of a function is the interior of \( \{ x \in M : u(x) = 0 \} \).

- Elements of \( C_c(\mathbb{R}^n) \) are continuous functions with compact support.

- \( C_0(\mathbb{R}^n) \) is complete in the sup norm. This can either be shown directly, or by showing that \( C_0(\mathbb{R}^n) \) is a closed subspace of \( C^*_b(\mathbb{R}^n) \).

- \( C_c(\mathbb{R}^n) \) is not complete. In fact, \( C_c(\mathbb{R}^n) \) is dense in \( C_0(\mathbb{R}^n) \). So \( C_0(\mathbb{R}^n) \) is a representation of the completion of \( C_c(\mathbb{R}^n) \) in the sup norm.
Series in Normed Linear Spaces

Let \((V, \| \cdot \|)\) be a normed linear space. Consider a series \(\sum_{n=1}^{\infty} v_n\) in \(V\).

**Definition.** We say the series *converges in* \(V\) if
\[
\exists v \in V \text{ such that } \lim_{N \to \infty} \|S_N - v\| = 0,
\]
where \(S_N = \sum_{n=1}^{N} v_n\) is the \(N\)th partial sum. We say this series *converges absolutely* if
\[
\sum_{n=1}^{\infty} \|v_n\| < \infty.
\]

**Caution:** Strictly speaking, if a series “converges absolutely” in a normed linear space, it does not have to converge in that space.

For example, the series \((1, 0, \cdots) + (0, \frac{1}{2}, 0, \cdots) + (0, 0, \frac{1}{4}, 0, \cdots)\) “converges absolutely” in \(\mathbb{F}_0^\infty\), but it doesn’t converge in \(\mathbb{F}_0^\infty\).

**Proposition.** A normed linear space \((V, \| \cdot \|)\) is complete iff every absolutely convergent series converges in \((V, \| \cdot \|)\).

**Proof Sketch**

\((\Rightarrow)\) Given an absolutely convergent series, show that the sequence of partial sums is Cauchy: for
\[
m > n, \quad \|S_m - S_n\| \leq \sum_{j=n+1}^{m} \|v_j\|.
\]

\((\Leftarrow)\) Given a Cauchy sequence \(\{x_n\}\), choose \(n_1, n_2 < \cdots\) inductively so that for
\[
k = 1, 2, \ldots, \ (\forall n, m \geq n_k) \quad \|x_n - x_m\| \leq 2^{-k}.
\]
Then \(\|x_{n_k} - x_{n_{k+1}}\| \leq 2^{-k}\). The series \(x_{n_1} + \sum_{k=2}^{\infty} (x_{n_k} - x_{n_{k-1}})\) is absolutely convergent. Let \(x\) be its limit. Then \(x_n \to x\).
Norms on Operators

If $V$, $W$ are vector spaces, then so is the space of linear transformations from $V$ to $W$ denoted

$$\mathcal{L}(V, W).$$

When $V = W$, $\mathcal{L}(V, V) = \mathcal{L}(V)$ is an algebra with composition as multiplication.

Norms on $\mathcal{L}(V)$ compatible with composition are particularly useful. A norm on $\mathcal{L}(V)$ is said to be \textit{submultiplicative} if

$$\|A \circ B\| \leq \|A\| \cdot \|B\| .$$

Not all matrix norms are submultiplicative.

For $A \in \mathbb{C}^{n \times n}$, define

$$\|A\| = \sup_{1 \leq i, j \leq n} |a_{ij}| .$$

Then, if

$$A = B = \begin{pmatrix} 1 & \cdots & 1 \\ & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

then $\|A\| = \|B\| = 1$, but $AB = A^2 = nA$ so $\|AB\| = n$.

But it can be shown that the norm

$$A \mapsto n \sup_{1 \leq i, j \leq n} |a_{ij}|$$

is submultiplicative.
Bounded Linear Operators

Let \((V, \| \cdot \|_v)\) and \((W, \| \cdot \|_w)\) be normed linear spaces.

An \(L \in \mathcal{L}(V, W)\) is called a \textit{bounded linear operator} if

\[
\sup_{\|v\|_v = 1} \|Lv\|_w < \infty.
\]

\(\mathcal{B}(V, W)\) is the set of all bounded linear operators from \(V\) to \(W\).

If \(W = \mathbb{F}\), we have \textit{bounded linear functionals}, and we set

\[
V^* = \mathcal{B}(V, \mathbb{F})
\]

If \(\dim V < \infty\), then \(\mathcal{L}(V, W) = \mathcal{B}(V, W)\), so also \(V^* = V'\).

Not all linear operators are bounded.
Let \(V = \mathcal{P}\) be the space of polynomials with norm

\[
\|p\| = \sup_{0 \leq x \leq 1} |p(x)|.
\]

Then \(\frac{d}{dx} : \mathcal{P} \rightarrow \mathcal{P}\) is not a bounded linear operator:

\[
\|x^n\| = 1 \quad \text{for all } n \geq 1 \quad \text{but} \quad \left\| \frac{d}{dx} x^n \right\| = \|nx^{n-1}\| = n
\]
Operator Norms

**Definition.** Let $L : V \rightarrow W$ be a bounded linear operator between normed linear spaces, i.e., $L \in \mathcal{B}(V, W)$. Define the operator norm of $L$ to be

$$||L|| = \sup_{||v|| \leq 1} ||Lv||_w$$

$\mathcal{B}(V, W)$ is a normed linear space.

In the special case $W = \mathbb{F}$, the norm

$$||f|| = \sup_{||v|| \leq 1} |f(v)|$$

on $V^*$ is called the **dual norm**.

Therefore,

$$|f(v)| \leq ||f|| ||v||, \quad \forall v \in V, \ f \in V^*.$$ 

If dim $V < \infty$, choose bases to identify $V$ and $V^*$ with $\mathbb{F}^n$. Thus every norm on $\mathbb{F}^n$ has a dual norm on $\mathbb{F}^n$. We sometimes write $\mathbb{F}^{n*}$ for $\mathbb{F}^n$ when it is being identified with $V^*$. 