Annihilators and Orthogonal Projections

Suppose $W \subset V$ is a subspace.

Define the orthogonal complement (read $W$ “perp”) 

$$W^\perp = \{ v \in V : \langle v, w \rangle = 0 \ (\forall \ w \in W) \}.$$

The orthogonal complement $W^\perp$ is a subspace of $V$.

The use of the same notation as used for the annihilator of $W$ is justified since the image of $W^\perp$ under the map $w \to w^*$ is precisely the annihilator of $W$.

If $\dim V < \infty$, a dimension count and the obvious $W \cap W^\perp = \{0\}$ show that 

$$V = W \oplus W^\perp.$$

So in a finite dimensional inner product space, a subspace $W$ determines a natural complement, namely $W^\perp$.

The induced projection onto $W$ (along $W^\perp$) is called the orthogonal projection onto $W$.
Norms

A norm on a vector space $V$ is a function $\|\cdot\| : V \to [0, \infty)$ satisfying

(i) $(\forall v \in V) \quad \|v\| \geq 0$, and $\|v\| = 0$ iff $v = 0$

(ii) $(\forall \alpha \in \mathbb{F})(\forall v \in V) \quad \|\alpha v\| = |\alpha| \cdot \|v\|$, and

(iii) (triangle inequality) $(\forall v, w \in V) \quad \|v + w\| \leq \|v\| + \|w\|.$

The pair $(V, \|\cdot\|)$ is called a normed linear space (or normed vector space).

Fact. A norm $\|\cdot\|$ on a vector space $V$ induces a metric $d$ on $V$ by $d(v, w) = \|v - w\|.$
Examples of Normed Linear Spaces

(1) $\ell^p$-norm on $\mathbb{F}^n$ ($1 \leq p \leq \infty$)

(a) $p = \infty$: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$, $x \in \mathbb{F}^n$

(b) $1 \leq p < \infty$: $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$, $x \in \mathbb{F}^n$.

The triangle inequality

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}$$

is known as “Minkowski’s inequality.” It is a consequence of Hölder’s inequality.

Integral versions of these inequalities are proved in real analysis texts, e.g., Folland or Royden. The proofs for vectors in $\mathbb{F}^n$ are analogous to the proofs for integrals.

(2) $\ell^p$-norm on $\ell^p$ (subspace of $\mathbb{F}^\infty$) ($1 \leq p \leq \infty$)

(a) $p = \infty$:

$$\ell^\infty = \{x \in \mathbb{F}^\infty : \sup_{i \geq 1} |x_i| < \infty\} \quad \|x\|_\infty = \sup_{i \geq 1} |x_i|$$

for $x \in \ell^\infty$.

(b) $1 \leq p < \infty$:

$$\ell^p = \left\{x \in \mathbb{F}^\infty : \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} < \infty\right\}, \quad \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$$

for $x \in \ell^p$. 

(3) $L^p$ norm on $C([a, b])$ \((1 \leq p \leq \infty)\)

(a) $p = \infty$: \[\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|.\]

Since $|f(x)|$ is a continuous, real-valued function on the compact set $[a, b]$, it takes on its maximum, so the "sup" is actually a "max" here:
\[\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|.\]

(b) $1 \leq p < \infty$: \[\|f\|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}.\]

Use continuity of $f$ to show that \[\|f\|_p = 0 \Rightarrow f(x) \equiv 0 \text{ on } [a, b].\]

The triangle inequality
\[\left( \int_a^b |f(x) + g(x)|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_a^b |g(x)|^p \, dx \right)^{\frac{1}{p}}\]

is Minkowski’s inequality, a consequence of Hölder’s inequality:
\[\int_a^b f(x)g(x) \, dx \leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}},\]

where \[\frac{1}{p} + \frac{1}{q} = 1.\]
Continuous Linear Operators on Normed Linear Spaces

**Theorem.**

$(V, \| \cdot \|_v)$ and $(W, \| \cdot \|_w)$ are normed linear spaces.

$L : V \to W$ is a linear transformation. Then the following are equivalent:

(a) $L$ is continuous

(b) $L$ is uniformly continuous \textit{(Lipschitz continuous)}

(c) $(\exists C')$ so that $(\forall v \in V)$ \hspace{1em} $\|Lv\|_w \leq C\|v\|_v$.

**Proof**

(a) $\Rightarrow$ (c):

Suppose $L$ is continuous. Then $L$ is continuous at $v = 0$.

Let $\epsilon = 1$. Then $\exists \delta > 0$ such that

$\|v\|_v \leq \delta \quad \Rightarrow \quad \|Lv\|_w \leq 1$

(as $L(0) = 0$). For any $v \neq 0$,

$\|\delta v/\|v\|_v\|_v \leq \delta \quad \Rightarrow \quad \|L(\delta v/\|v\|_v)\|_w \leq 1$,

i.e., $\|Lv\|_w \leq \frac{1}{\delta}\|v\|_v$. Let $C = \frac{1}{\delta}$.

(c) $\Rightarrow$ (b):

Condition (c) implies that

$(\forall v_1, v_2 \in V) \quad \|Lv_1 - Lv_2\|_w = \|L(v_1 - v_2)\|_w \leq C\|v_1 - v_2\|_v$.

Hence $L$ is uniformly continuous (given $\epsilon$, let $\delta = \frac{\epsilon}{C}$, etc.). In fact, $L$ is uniformly Lipschitz continuous with Lipschitz constant $C$.

(b)$\Rightarrow$(a) is immediate.
Bounded Linear Operators and Their Norms

**Definition** \( V \) and \( W \) are normed linear spaces. 
\( L : V \to W \) a linear operator. 

If 
\[
\sup_{v \in V, v \neq 0} \frac{\|Lv\|_w}{\|v\|_v} < \infty,
\]

then \( L \) is called a **bounded linear operator** from \( V \) to \( W \).

**Remarks.**

(1) Note that it is the *norm ratio*
\[
\frac{\|Lv\|_w}{\|v\|_v}
\]
(or “stretching factor”) that is bounded, *not* \( \{\|Lv\|_w : v \in V\} \).

(2) The theorem above says that if \( V \) and \( W \) are normed linear spaces and \( L : V \to W \) is linear, then

\[
L \text{ is continuous} \iff L \text{ is uniformly continuous} \iff L \text{ is a bounded linear operator}.
\]

**Definition** \( V \) and \( W \) are normed linear spaces. 
\( L : V \to W \) is a bounded linear operator. 

Define the *operator norm* of \( L \) to be 
\[
\|L\| = \sup_{v \in V, v \neq 0} \frac{\|Lv\|_w}{\|v\|_v}.
\]
**Equivalence of Norms**

It is easily seen that the norm in a normed linear space is a continuous mapping from the space into \( \mathbb{R} \).

This follows from the *other half* of the triangle inequality:

\[
|\|u\| - \|v\| | \leq \|u - v\|.
\]

**Definition** Two norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \), both on the same vector space \( V \), are called *equivalent norms* on \( V \) if \( \exists \) constants \( C_1, C_2 > 0 \) for which

\[
(\forall v \in V) \quad \frac{1}{C_1}\|v\|_2 \leq \|v\|_1 \leq C_2\|v\|_2.
\]

**Fact.**

\( V \) is finite-dim if and only if any two norms on \( V \) are equivalent.

**Remarks.**

(1) All norms on a fixed finite dimensional vector space are equivalent. But be careful when studying problems where there is a sequence of finite dimensional spaces of increasing dimensions: the constants \( C_1 \) and \( C_2 \) in the equivalence can depend on the dimension. For example, in \( \mathbb{F}^n \)

\[
\|x\|_2 \leq \sqrt{n}\|x\|_\infty.
\]

(2) In a normed linear space \( V \), the closed unit ball

\[
\{v \in V : \|v\| \leq 1\}
\]

is compact iff \( \dim V < \infty \).
Examples

(1) Set

\[ \mathbb{F}_0^\infty = \{ x \in \mathbb{F}^\infty : (\exists N)(\forall n \geq N) \ x_n = 0 \}. \]

For \( 1 \leq p < q \leq \infty \), the \( \ell^p \) and \( \ell^q \) norms are not equivalent.

Consider \( p = 1, q = \infty \).

Note that

\[ \|x\|_\infty \leq \sum_{i=1}^{\infty} |x_i| = \|x\|_1. \]

But if

\[ y_1 = (1,0,0,\cdots), \ y_2 = (1,1,0,\cdots), \ y_3 = (1,1,1,0,\cdots), \ldots \]

then

\[ \|y_n\|_\infty = 1 \quad \text{and} \quad \|y_n\|_1 = n \quad \forall n. \]

So there does not exist a constant \( C \) for which

\[ (\forall x \in \mathbb{F}_0^\infty) \quad \|x\|_1 \leq C \|x\|_\infty. \]

(2) In \( \ell^2 \) (a subspace of \( \mathbb{F}^\infty \)) with norm

\[ \|x\|_2 = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}, \]

the closed unit ball

\[ \{ x \in \ell^2 : \|x\|_2 \leq 1 \} \]

is not compact. The sequence

\[ e_1, e_2, e_3, \ldots \]

is bounded, \( \|e_i\|_{\ell^2} \leq 1 \), and all are in the closed unit ball, but no subsequence can converge because

\[ \|e_i - e_j\|_{\ell^2} = \sqrt{2} \quad \text{for} \quad i \neq j. \]
Norms induced by inner products

Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space. Define

\[ ||v|| = \sqrt{\langle v, v \rangle} . \]

We have

\[ ||v|| \geq 0 \quad \text{with} \quad ||v|| = 0 \iff v = 0 , \]

and

\[ (\forall \alpha \in \mathbb{F})(\forall v \in V) \quad ||\alpha v|| = |\alpha| \cdot ||v|| . \]

To show that \( || \cdot || \) is a norm we need the triangle inequality.

The Cauchy-Schwarz inequality

For all \( v, w \in V \) we have

\[ |\langle v, w \rangle| \leq ||v|| \cdot ||w|| , \]

with equality iff \( v \) and \( w \) are linearly dependent.

**Proof:**

Case (i) If either \( v = 0 \) or \( w = 0 \), done.

Case (ii) If \( ||v|| = ||w|| = 1 \) and \( \langle v, w \rangle \geq 0 \), then

\[ 0 \leq ||v - w||^2 = \langle v - w, v - w \rangle = \langle v, v \rangle - 2\Re \langle v, w \rangle + \langle w, w \rangle = 2(1 - \langle v, w \rangle) \]

so \( \langle v, w \rangle \leq 1 \) (with equality iff \( v = w \)).

Case (iii) For \( v \neq 0 \neq w \), pick \( \alpha \in \mathbb{F} \) with \( |\alpha| = 1 \) and \( \alpha \langle v, w \rangle \geq 0 \).

Let \( v_1 = \frac{\alpha}{||v||} v \) and \( w_1 = \frac{w}{||w||} \).

Then \( ||v_1|| = ||w_1|| = 1 \) and \( \langle v_1, w_1 \rangle \geq 0 \), so

\[ \frac{|\langle v, w \rangle|}{||v|| \cdot ||w||} = \frac{\alpha \langle v, w \rangle}{||v|| \cdot ||w||} = \frac{\langle v_1, w_1 \rangle}{||v_1|| \cdot ||w_1||} \leq 1 \]

(with equality iff \( v_1 = w_1 \)).
The triangle inequality follows:

\[
\|v + w\|^2 = \langle v + w, v + w \rangle
\]
\[
= \langle v, v \rangle + 2\Re\langle v, w \rangle + \langle w, w \rangle
\]
\[
\leq \|v\|^2 + 2|\langle v, w \rangle| + \|w\|^2
\]
\[
\leq \|v\|^2 + 2\|v\| \cdot \|w\| + \|w\|^2
\]
\[
= (\|v\| + \|w\|)^2.
\]

So \(\|v\| = \sqrt{\langle v, v \rangle}\) is a norm on \(V\).

It is called the norm induced by the inner product \(\langle \cdot, \cdot \rangle\).

An inner product induces a norm, which induces a metric

\[(V, \langle \cdot, \cdot \rangle) \leftrightarrow (V, \| \cdot \|) \leftrightarrow (V, d)\].\]
**Examples.**

(1) The Euclidean norm [i.e. $\ell^2$ norm] on $\mathbb{F}^n$ is induced by the standard inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} : \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} |x_i|^2}.$$ 

(2) Let $A \in \mathbb{F}^{n \times n}$ be Hermitian-symmetric and positive definite, and let

$$\langle x, y \rangle_A = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} \overline{y_j} \text{ for } x, y \in \mathbb{F}^n.$$ 

Then $\langle \cdot, \cdot \rangle_A$ is an inner product on $\mathbb{F}^n$, which induces the norm

$$\|x\|_A = \sqrt{\langle x, x \rangle_A} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} \overline{x_j} = \sqrt{x^T Ax} = \sqrt{x^H A x}}.$$ 

(3) The $\ell^2$-norm on $\ell^2$ (subspace of $\mathbb{F}^\infty$) is induced by the inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} : \|x\|_2 = \sqrt{\sum_{i=1}^{\infty} x_i \overline{x_i} = \sum_{i=1}^{\infty} |x_i|^2}.$$ 

(4) The $L^2$ norm

$$\|u\|_2 = \left( \int_{a}^{b} |u(x)|^2 \, dx \right)^{\frac{1}{2}},$$

on $C([a, b])$ is induced by the inner product

$$\langle u, v \rangle = \int_{a}^{b} u(x) \overline{v(x)} \, dx.$$
The Closed Unit Balls

For $\ell^p$ norms in $\mathbb{R}^2$ ($1 \leq p \leq \infty$)

\[ p = \infty \quad p = 2 \quad p = 1 \]

Convexity
A subset $C$ of a vector space $V$ is called convex if
\[
(\forall v, w \in C)(\forall t \in [0, 1]) \quad tv + (1 - t)u \in C.
\]

Let $B = \{v \in V : \|v\| \leq 1\}$ denote the closed unit ball in a finite dimensional normed linear space.

**Facts.**

1. $B$ is convex.

2. $B$ is compact.

3. $B$ is symmetric
   
   (if $v \in B$ and $\alpha \in \mathbb{F}$ with $|\alpha| = 1$, then $\alpha v \in B$).

4. The origin is in the interior of $B$.

**Lemma.** If $\dim V < \infty$ and $B \subset V$ satisfies the four conditions above, then there is a unique norm on $V$ for which $B$ is the closed unit ball:

\[
\|v\| = \inf\{c > 0 : \frac{v}{c} \in B\}.
\]