Annihilators

Let $S \subset V$.

Define the annihilator $S^\perp \subset V'$ by

$$S^\perp = \{ f \in V' : (\forall v \in S) \, f(v) = 0 \}.$$  

Clearly

$$S^\perp = (\text{Span}(S))^\perp ,$$

and

$$S^{\perp \perp} \subset V'' .$$

**Proposition.** If $\dim V < \infty$, then $S^{\perp \perp} = \text{Span}(S)$.

*Proof.* As above we make the identification $V = V''$ and so

$$\text{Span}(S) \subset S^{\perp \perp} .$$

For the reverse, we assume WLOG that $S$ is a subspace with basis

$$\{s_1, \ldots, s_m\} .$$

Complete to a basis of $V$,

$$\{s_1, \ldots, s_{m+1}, \ldots, s_n\} .$$

Then the dual basis vectors $\{f_{m+1}, \ldots, f_n\}$ are a basis for $S^\perp$. So

$$\dim S^{\perp \perp} = n - \dim S^\perp = n - (n - \dim S) = \dim S .$$

Since $S \subset S^{\perp \perp}$, the proof is complete.  \[\square\]
The Fundamental Theorem of the Alternative

**Proposition.** Suppose $L \in \mathcal{L}(V, W)$. Then $\mathcal{N}(L^*) = \mathcal{R}(L)^\perp$.

**Proof.** Clearly both are subspaces of $W'$. Let $g \in W'$. Then

$$g \in \mathcal{N}(L^*) \iff L^*g = 0$$

$$\iff (\forall v \in V) \ (L^*g)(v) = 0$$

$$\iff (\forall v \in V) \ g(Lv) = 0$$

$$\iff g \in \mathcal{R}(L)^\perp. \quad \Box$$

The result is called the *Fundamental Theorem of the Alternative* since it is equivalent to the following:

One of the two alternatives (A) and (B) must hold, and both (A) and (B) cannot hold.

(A) \[
\begin{bmatrix}
\text{The system} \\
\quad y = Lx \\
\text{is solvable.}
\end{bmatrix}
\]

(B) \[
\begin{bmatrix}
\text{There exists } w \in W' \text{ such that} \\
\quad L^*w = 0 \\
\quad \text{and } w(y) \neq 0.
\end{bmatrix}
\]
Bilinear Forms

A function $\varphi : V \times V \to \mathbb{F}$ is called a \textit{bilinear form} if it is linear in each variable separately:

$$\varphi \left( \sum x_i v_i \sum y_j v_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \varphi(v_i, v_j).$$

\text{Examples:}

(1) For $A \in \mathbb{F}^{n \times n}$, the function

$$\varphi(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j$$

is a bilinear form.

In fact, all bilinear forms on $\mathbb{F}^n$ are of this form, as

$$\varphi \left( \sum x_i e_i \sum y_j e_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \varphi(e_i, e_j),$$

so just set $a_{ij} = \varphi(e_i, e_j)$.

In general, let $V$ be finite-dimensional with basis $\{v_1, \ldots, v_n\}$. Let $v \in V$ with $v = \sum x_i v_i$ and $w = \sum y_j v_j$.

If $\varphi$ is a bilinear form on $V$, then

$$\varphi(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \varphi(v_i, v_j) = x^T A y,$$

where $A \in \mathbb{F}^{n \times n}$ satisfies $a_{ij} = \varphi(v_i, v_j)$.

$A$ is called the \textit{matrix of }$\varphi$ with respect to the basis $\{v_1, \ldots, v_n\}$. 
(2) One can also use infinite matrices \((a_{ij})_{i,j \geq 1}\) for \(V = \mathbb{F}^{\infty}\) as long as convergence conditions are imposed.

For example, if all \(|a_{ij}| \leq M\), then

\[
\varphi(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} x_i y_j
\]

defines a bilinear form on \(\ell^1\) since

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij} x_i y_j| \leq M \left( \sum_{i=1}^{\infty} |x_i| \right) \left( \sum_{j=1}^{\infty} |y_j| \right).
\]

(3) If \(f, g \in V'\), then \(\varphi(x, y) = f(x)g(y)\) is a bilinear form.

(4) If \(V = C[a, b]\), then

(i) for \(k \in C([a, b] \times [a, b])\), \(\int_a^b \int_a^b k(x, y)u(x)v(y)\,dx\,dy\)

(ii) for \(h \in C([a, b])\), \(\int_a^b h(x)u(x)v(x)\,dx\)

(iii) for \(x_0 \in [a, b]\), \(u(x_0) \int_a^b v(x)\,dx\)

are all examples of bilinear forms.

A bilinear form is **symmetric** if

\[
(\forall v, w \in V) \quad \varphi(v, w) = \varphi(w, v).
\]

In the finite-dimensional case, this implies the matrix \(A\) be symmetric (wrt any basis), i.e.,

\[
A = A^T, \quad \text{or} \quad (\forall i, j) \ a_{ij} = a_{ji}.
\]
Sesquilinear Forms

Let $\mathbb{F} = \mathbb{C}$, a sesquilinear form on $V$, $\varphi : V \times V \to \mathbb{C}$, is linear in the first variable and conjugate-linear in the second variable, i.e.,

$$\varphi(v, \alpha_1 w_1 + \alpha_2 w_2) = \bar{\alpha}_1 \varphi(v_1, w_1) + \bar{\alpha}_2 \varphi(v, w_2).$$

On $\mathbb{C}^n$ all sesquilinear forms are of the form

$$\varphi(z, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i \bar{w}_j$$

for some $A \in \mathbb{C}^{n \times n}$.

To be able to discuss bilinear forms over $\mathbb{R}$ and sesquilinear forms over $\mathbb{C}$ at the same time, we will speak of a sesquilinear form over $\mathbb{R}$ and mean just a bilinear form over $\mathbb{R}$.

A sesquilinear form is said to be Hermitian-symmetric (or sometimes just Hermitian) if

$$(\forall v, w \in V) \quad \varphi(v, w) = \overline{\varphi(w, v)}$$

(when $\mathbb{F} = \mathbb{R}$, we say the form is symmetric).

This corresponds to the condition that $A = A^H$ where

$$A^H = \overline{A^T} \quad \text{i.e.,} \quad (A^H)_{ij} = \overline{(A^T)_{ji}}.$$ 

$A^H$ is the Hermitian transpose (or conjugate transpose) of $A$ when $\mathbb{F} = \mathbb{C}$.

If $A = A^H \in \mathbb{C}^{n \times n}$, we say $A$ is Hermitian (-symmetric).

If $A = A^T \in \mathbb{R}^{n \times n}$ and $\mathbb{F} = \mathbb{R}$, we say $A$ is symmetric.
Inner Products

Associate the quadratic form \( \varphi(v, v) \) with the sesquilinear form \( \varphi \). We say that \( \varphi \) is nonnegative (or positive semi-definite) if

\[
(\forall v \in V) \quad \varphi(v, v) \geq 0,
\]

and that \( \varphi \) is positive (or positive definite) if

\[
(\forall v \in V, v \neq 0) \quad \varphi(v, v) > 0.
\]

By an inner product on \( V \), we will mean a positive-definite Hermitian-symmetric sesquilinear form.

inner product

= positive-definite Hermitian-symmetric sesquilinear form

Examples: \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{R} \)

(1) \( \mathbb{F}^n \) with the Euclidean inner product

\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} = y^\textsc{h} x.
\]

(2) Let \( V = \mathbb{F}^n \) and \( A \in \mathbb{F}^{n \times n} \) be Hermitian-symmetric Define

\[
\langle x, y \rangle_A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i \overline{y_j} = y^\textsc{h} Ax.
\]

The requirement that \( \langle x, x \rangle_A > 0 \) for \( x \neq 0 \) so that \( \langle \cdot, \cdot \rangle_A \) is an inner product serves to define positive-definite matrices.
(3) Let $V$ be any finite-dimensional vector space. Choose a basis and thus identify $V \cong \mathbb{F}^n$. Transfer the Euclidean inner product to $V$ in the coordinate of this basis. The resulting inner product depends on the choice of basis. With respect to the coordinates induced by a basis, any inner product on a finite-dimensional vector space $V$ is of the form described in example (2) above.

(4) One can define an inner product on $\ell^2$ by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i.$$ 

To see that this sum converges absolutely, apply the finite-dimensional Cauchy-Schwarz inequality to obtain

$$\sum_{i=1}^{n} \left| x_i y_i \right| \leq \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |y_i|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |y_i|^2 \right)^{\frac{1}{2}}.$$ 

Now let $n \to \infty$ to deduce that the series $\sum_{i=1}^{\infty} x_i y_i$ converges absolutely.

(5) The $L^2$-inner product on $C([a, b])$ is given by

$$\langle u, v \rangle = \int_{a}^{b} u(x) \overline{v(x)} dx.$$
Inner Products and $V'$

- An inner product on $V$ determines an injection $V \rightarrow V'$.

For $w \in V$, define

$$w \mapsto w^* \in V' \quad \text{by} \quad w^*(v) = \langle v, w \rangle.$$

Since $w^*(w) = \langle w, w \rangle$ it follows that

$$w^* = 0 \Rightarrow w = 0,$$

so the map $w \mapsto w^*$ is injective (one to one).

- The map $w \mapsto w^*$ is conjugate-linear since

$$(\alpha w)^* = \bar{\alpha} w^*.$$

It is linear if $\mathbb{F} = \mathbb{R}$.

- The image of the mapping $w \mapsto w^*$ is a subspace of $V'$.

If $\dim V < \infty$, then this map is surjective too since $\dim V = \dim V'$.

In general, the mapping $w \mapsto w^*$ is not surjective.
Let $\dim V < \infty$ with inner product $\langle \cdot, \cdot \rangle$.

Choose a basis $\mathcal{B}$ and let $v, w \in V$ have coordinates in $\mathbb{F}^n$ given by

$$
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n
\end{pmatrix},
$$

respectively.

Let $A \in \mathbb{F}^{n \times n}$ be the inner product matrix in this basis, then

$$
w^*(v) = \langle v, w \rangle = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} y_j \right) x_i.
$$

It follows that $w^*$ has components

$$
b_i = \sum_{j=1}^n a_{ij} y_j
$$

with respect to the dual basis.

Therefore, the map $w \mapsto w^*$ corresponds to a mapping of its coordinates in the basis $\mathcal{B}$ to its coordinates in the dual basis $\mathcal{B}'$ given by the matrix-vector product

$$
\begin{pmatrix}
  b_1 \\
  \vdots \\
  b_n
\end{pmatrix} = A \begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n
\end{pmatrix}.
$$