Main Uniqueness Theorem

Before stating our main uniqueness result, we introduce a local form of Lipschitz continuity of the function \( f(t, x) \) in the \( x \) argument.

**Definition.** Let \( \mathcal{D} \) be an open set in \( \mathbb{R} \times \mathbb{F}^n \). We say that \( f(t, x) \) mapping \( \mathcal{D} \) into \( \mathbb{F}^n \) is *locally Lipschitz continuous with respect to \( x \) if*

\[
\forall (t_1, x_1) \in \mathcal{D}, \quad \exists \quad \alpha > 0, \quad r > 0 \quad \text{and} \quad L > 0
\]

for which

\[
[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D}
\]

and

\[
(\forall t \in [t_1 - \alpha, t_1 + \alpha]) \quad (\forall x, y \in \overline{B_r(x_1)}) \quad |f(t, x) - f(t, y)| \leq L|x - y|,
\]

i.e., \( f \) is uniformly Lipschitz continuous with respect to \( x \) in

\[
[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)}.
\]

We say \( f \in (C, \text{Lip}_{\text{loc}}) \) (not a standard notation) on \( \mathcal{D} \) if \( f \) is continuous on \( \mathcal{D} \) and locally Lipschitz continuous wrt \( x \) on \( \mathcal{D} \).

**Example.** Let \( \mathcal{D} \) be an open set in \( \mathbb{R} \times \mathbb{F}^n \). Suppose \( f(t, x) \) maps \( \mathcal{D} \) into \( \mathbb{F}^n \), \( f \) is continuous on \( \mathcal{D} \), and

\[
\text{for} \quad 1 \leq i, j \leq n, \quad \frac{\partial f_i}{\partial x_j} \quad \text{exists and is continuous in} \quad \mathcal{D},
\]

i.e., \( f \) is continuous on \( \mathcal{D} \) and \( C^1 \) with respect to \( x \) on \( \mathcal{D} \). Then \( f \in (C, \text{Lip}_{\text{loc}}) \) on \( \mathcal{D} \).
Main Uniqueness Theorem

Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$, and suppose

(a) $f \in (C, \text{Lip}_{loc})$ on $\mathcal{D}$,

(b) $(t_0, x_0) \in \mathcal{D},$

(c) $I \subset \mathbb{R}$ is an interval containing $t_0$
   (which may be open or closed at either end), and

(d) $x(t)$ and $y(t)$ are both solutions of the IVP

$$x' = f(t, x); \quad x(t_0) = x_0 \quad \text{in} \quad C^1(I)$$

which satisfy

$$(t, x(t)) \in \mathcal{D} \quad \text{and} \quad (t, y(t)) \in \mathcal{D} \quad \forall \ t \in I.$$ 

Then $x(t) \equiv y(t)$ on $I$. 
Proof
We first show $x(t) \equiv y(t)$ on $\{ t \in I : t \geq t_0 \}$.
If not, let

$$t_1 = \inf \{ t \in I : t \geq t_0 \text{ and } x(t) \neq y(t) \}.$$  

Then $x(t) = y(t)$ on $[t_0, t_1)$ so by continuity $x(t_1) = y(t_1)$
(if $t_1 = t_0$, this is obvious).
By continuity and the openness of $\mathcal{D}$ (as $(t_1, x(t_1)) \in \mathcal{D}$),

$$\exists \quad \alpha > 0 \quad \text{and} \quad r > 0$$

such that

$$[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D},$$

$f$ is uniformly Lipschitz continuous with respect to $x$ in

$$[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)},$$

and

$$x(t) \in \overline{B_r(x_1)} \quad \text{and} \quad y(t) \in \overline{B_r(x_1)} \quad \forall t \in I \cap [t_1 - \alpha, t_1 + \alpha].$$

By the previous theorem, $x(t) \equiv y(t)$ in $I \cap [t_1 - \alpha, t_1 + \alpha]$, contradicting the definition of $t_1$.
Hence

$$x(t) \equiv y(t) \quad \text{on} \quad \{ t \in I : t \geq t_0 \}.$$  

Similarly,

$$x(t) \equiv y(t) \quad \text{on} \quad \{ t \in I : t \leq t_0 \}.$$  

Hence $x(t) \equiv y(t)$ on $I$.  \qed

Remark. $t_0$ is allowed to be the left or right endpoint of $I$. 
Comparison Theorem for
Nonlinear Real Scalar Equations

**Theorem.** Let \( n = 1, \mathbb{F} = \mathbb{R} \), and suppose \( f(t, u) \) is continuous in \( t \) and Lipschitz continuous in \( u \).
Assume \( u(t), v(t) \) are \( C^1 \) for \( t \geq t_0 \) (on an interval \([t_0, b)\) or \([t_0, b]\)) and satisfy

\[
    u'(t) \leq f(t, u(t)), \quad v'(t) = f(t, v(t))
\]

and \( u(t_0) \leq v(t_0) \). Then

\[
    u(t) \leq v(t) \quad \text{for} \quad t \geq t_0.
\]

**Proof.** If to the contrary \( u(T) > v(T) \) for some \( T > t_0 \), then set

\[
    t_1 = \sup\{t : t_0 \leq t < T \quad \text{and} \quad u(t) \leq v(t)\}.
\]

Then

\[
    t_0 \leq t_1 < T, \quad u(t_1) = v(t_1), \quad \text{and} \quad u(t) > v(t) \quad \text{for} \quad t_1 < t \leq T
\]
(by continuity of \( u - v \)). For

\[
    t_1 \leq t \leq T, \quad |u(t) - v(t)| = u(t) - v(t),
\]
so we have

\[
    (u - v)' \leq f(t, u) - f(t, v) \leq L|u - v| = L(u - v).
\]

By Gronwall’s inequality applied to \( u - v \) on \([t_1, T]\), with

\[
    (u - v)(t_1) = 0, \quad a(t) \equiv L, \quad b(t) \equiv 0,
\]
\( (u - v)(t) \leq 0 \) on \([t_1, T]\), a contradiction. \( \square \)
Remarks.

(1) As with the differential form of Gronwall’s inequality a solution of the differential inequality \( u' \leq f(t, u) \) is bounded above by the solution of the equality (i.e., the DE \( v' = f(t, v) \)).

(2) It can be shown under the same hypotheses that if \( u(t_0) < v(t_0) \), then \( u(t) < v(t) \) for \( t \geq t_0 \) (problem 4 on Prob. Set 1).

(3) Caution: It may happen that \( u'(t) > v'(t) \) for some \( t \geq t_0 \): \( u(t) \leq v(t) \not\Rightarrow u'(t) \leq v'(t) \).

Corollary. Let \( n = 1, \mathbb{F} = \mathbb{R} \). Suppose \( f(t, u) \leq g(t, u) \) are continuous in \( t \) and \( u \), and one of them is Lipschitz continuous in \( u \). Suppose also that \( u(t), v(t) \) are \( C^1 \) for \( t \geq t_0 \) (on \( [t_0, b) \) or \( [t_0, b] \)) and satisfy

\[ u' = f(t, u), \quad v' = g(t, v), \quad \text{and} \quad u(t_0) \leq v(t_0). \]

Then

\[ u(t) \leq v(t) \quad \text{for} \quad t \geq t_0. \]

Proof. Suppose first that \( g \) satisfies the Lipschitz condition. Then

\[ u' = f(t, u) \leq g(t, u). \]

Now apply the theorem. If \( f \) satisfies the Lipschitz condition, apply the first part of this proof to

\[ \tilde{u}(t) \equiv -v(t), \quad \tilde{v}(t) \equiv -u(t), \quad \tilde{f}(t, u) = -g(t, -u), \quad \tilde{g}(t, u) = -f(t, -u). \]

Remark. Again, if \( u(t_0) < v(t_0) \), then \( u(t) < v(t) \) for \( t \geq t_0 \).
Continuation of Solutions in Time

We consider two kinds of results

- *local continuation* (no Lipschitz condition on \( f \))
- *global continuation* (for locally Lipschitz \( f \))

Local Continuation (Continuation at a Point)

Assume \( x(t) \) is a solution of the DE \( x' = f(t, x) \) on an interval \( I \) and \( f \) is continuous on a subset \( S \subset \mathbb{R} \times \mathbb{F}^n \) containing \( \left\{ (t, x(t)) : t \in I \right\} \).

Note: no Lipschitz condition is assumed.

**Case 1:** *I is closed at the right end,*
i.e., \( I = (-\infty, b], [a, b], \text{ or } (a, b] \).
Assume further that \( (b, x(b)) \) is in the interior of \( S \). Then the solution can be extended (by Cauchy-Peano) to an interval with right end \( b + \beta \) for some \( \beta > 0 \). This is done by solving the IVP

\[
x' = f(t, x) \quad \text{with initial value } x(b) \text{ at } t = b
\]
on an interval \([b, b + \beta] \). To show that the continuation is \( C^1 \) at \( t = b \), note that the extended \( x(t) \) satisfies the integral equation

\[
x(t) = x(b) + \int_{b}^{t} f(s, x(s)) \, ds
\]
on \( I \cup [b, b + \beta] \).
**Case 2:** $I$ is open at the right end, i.e., $I = (-\infty, b)$, $[a, b)$, or $(a, b)$ with $b < \infty$.

Assume further that $f(t, x(t))$ is **bounded** on $[t_0, b)$ for some $t_0 < b$ with $[t_0, b) \subseteq I$, say $|f(t, x(t))| \leq M$ on $[t_0, b)$.

In this case the integral equation

\[
(*) \quad x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s)) \, ds
\]

holds for $t \in I$. In particular, for $t_0 \leq \tau \leq t < b$,

\[
|x(t) - x(\tau)| = \left| \int_{\tau}^{t} f(s, x(s)) \, ds \right| \leq \int_{\tau}^{t} |f(s, x(s))| \, ds \leq M|t - \tau|.
\]

Thus, for any sequence $t_n \uparrow b$, \(\{x(t_n)\}\) is Cauchy. This implies $\lim_{t \to b^-} x(t)$ exists; call it $x(b^-)$. So $x(t)$ has a continuous extension from $I$ to $I \cup \{b\}$.

- If in addition $(b, x(b^-))$ is in $\mathcal{S}$, then $(*)$ holds on $I \cup \{b\}$ as well, so $x(t)$ is a $C^1$ solution of $x' = f(t, x)$ on $I \cup \{b\}$.

- Finally, if $(b, x(b^-))$ is in the interior of $\mathcal{S}$, we are back in Case 1 and can extend the solution $x(t)$ beyond $t = b$.

- The assumption that $f(t, x(t))$ is bounded on $[t_0, b)$ can be restated with a slightly different emphasis: for some $t_0 \in I$, $\{(t, x(t)) : t_0 \leq t < b\}$ stays within a subset of $\mathcal{S}$ on which $f$ is bounded. For example, if $\{(t, x(t)) : t_0 \leq t < b\}$ stays within a compact subset of $\mathcal{S}$, this condition is satisfied.

**Case 3:** $I$ is closed at the left end — similar to Case 1.

**Case 4:** $I$ is open at the left end — similar to Case 2.
Global Continuation

Assume $f(t, x)$ is continuous on an open set $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$ and is locally Lipschitz continuous with respect to $x$ on $\mathcal{D}$. Write $f \in (C, \text{Lip}_{\text{loc}})$ on $\mathcal{D}$.

Let $(t_0, x_0) \in \mathcal{D}$ and consider the IVP

$$x' = f(t, x), \quad x(t_0) = x_0.$$ 

It has been shown that a unique solutions exist on both $[t_0, t_0 + \alpha_+]$ and $(-\alpha_- + t_0, t_0]$, and that this gives a unique solution on $(-\alpha_- + t_0, \alpha_+)$ for some $\alpha_+, \alpha_- > 0$.

Set

$$T_+ = \sup \{ t > t_0 : \exists \text{ a solution of IVP on } [t_0, t) \}, \quad \text{and} \quad T_- = \inf \{ t < t_0 : \exists \text{ a solution of IVP on } (t, t_0) \}.$$ 

$(T_-, T_+)$ is the maximal interval of existence of the solution of the IVP. It is possible that $T_+ = \infty$ and/or $T_- = -\infty$.

The maximal interval $(T_-, T_+)$ must be open: if the solution could be extended to $T_+$ (or $T_-$), this would contradict the local continuation results since $\mathcal{D}$ is open.

Ideally, $T_+ = +\infty$ and $T_- = -\infty$.

Another possibility is if $f(t, x)$ is not defined for $t \geq T_+$. For example, if $a(t) = \frac{1}{1-t}$, and $x'(t) = a(t)$. Here we don’t expect the solution to exist beyond $t = 1$.

But less desirable behavior can occur.

For example, for the IVP: $x^1 = x^2, \ x(0) = x_0 > 0, \ t_0 = 0$, and $\mathcal{D} = \mathbb{R} \times \mathbb{R}$. The solution $x(t) = (x_0^{-1} - t)^{-1}$ blows up at $T_+ = 1/x_0$ (note that $T_- = -\infty$). Observe that $x(t) \to \infty$ as $t \to (T_+)^-$. So the solution does not just “stop” in the interior of $\mathcal{D}$.

This kind of blow-up behavior must occur if a solution cannot be continued to the whole real line.
**Theorem.** (Solution Blow-Up)
Suppose \( f \in (C, \text{Lip}_{\text{loc}}) \) on an open set \( \mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n \). Let \( (t_0, x_0) \in \mathcal{D} \), and let \( (T_-, T_+) \) be the maximal interval of existence of the solution of the IVP
\[
x' = f(t, x), \quad x(t_0) = x_0.
\]
If \( T_+ < +\infty \) (\( T_- > -\infty \)), then for any compact set \( K \subset \mathcal{D} \), there exists a \( T < T_+ \) (\( T_- < T \)) for which \( (t, x(t)) \notin K \) for \( t > T \) (\( t < T \)).

**Proof.** If not, \( \exists t_j \to T_+ \) with \( (t_j, x(t_j)) \in K \) for all \( j \). By taking a subsequence, we may assume that \( x(t_j) \) also converges to \( x_+ \in \mathbb{F}^n \), and
\[
(t_j, x(t_j)) \to (T_+, x_+) \in K \subset \mathcal{D}.
\]
We can thus choose \( r > 0, \tau > 0, N \in \mathbb{N} \) such that
\[
\mathcal{S} = \bigcup_{j=N}^{\infty} \{(t, x) : |t - t_j| \leq \tau, |x - x(t_j)| \leq r\} \subset \mathcal{D}.
\]
Since \( \mathcal{D} \) is compact, there is an \( M \) for which \( |f(t, x)| \leq M \) on \( \mathcal{S} \). By the local existence theorem, the solution of \( x' = f(t, x) \) starting at the initial point \( (t_j, x(t_j)) \) exists for a time interval of length
\[
T' \equiv \min \left\{ \tau, \frac{r}{M} \right\},
\]
*independent of \( i \).* Choose \( j \) for which \( t_j > t_+ - T' \). Then \( (t, x(t)) \) exists in \( \mathcal{D} \) beyond time \( T_+ \), which is a contradiction. \( \square \)
Autonomous Systems

The ODE \( x'(t) = f(t, x) \) is called an autonomous system if \( f(t, x) \) is independent of \( t \), i.e., the ODE is of the form

\[
x' = f(x).
\]

**Remarks.**

(1) Time translates of solutions of an autonomous system are again solutions:

\( x(t) \) a solution \( \implies \) \( x(t - c) \) is a solution for any constant \( c \).

(2) Any ODE \( x' = f(t, x) \) is equivalent to an autonomous system. Define “\( x_{n+1} = t \)” and set

\[
\tilde{x} = (x_{n+1}, x) \in \mathbb{F}^{n+1}
\]

\[
\tilde{x}' = \tilde{f}(\tilde{x}) = \tilde{f}(x_{n+1}, x) = \begin{bmatrix} 1 \\ f(x_{n+1}, x) \end{bmatrix} \in \mathbb{F}^{n+1}
\]

and consider the autonomous IVP

\[
\tilde{x}' = \tilde{f}(\tilde{x}), \quad \tilde{x}(t_0) = \begin{bmatrix} t_0 \\ x_0 \end{bmatrix}.
\]

This IVP is equivalent to the IVP

\[
x' = f(t, x), \quad x(t_0) = x_0.
\]
Continuation for Autonomous Systems

Suppose $f(x)$ is defined and locally Lipschitz continuous on an open set $\mathcal{U} \subset \mathbb{R}^n$. Take $\mathcal{D} = \mathbb{R} \times \mathcal{U}$. Suppose $T_+ < \infty$ and $C$ is a compact subset of $\mathcal{U}$. Take $K = [t_0, T_+] \times C$ in the ODE Blow-Up Theorem.

Then

$$\exists T < T_+ \text{ such that } x(t) \notin C \text{ for } T < t < T_+.$$

In this case we say that

$$x(t) \to \partial \mathcal{U} \cup \{\infty\} \text{ as } t \to (T_+)^-,$$

meaning that

$$(\forall C^{\text{compact}} \subset \mathcal{U})(\exists T < T_+) \text{ such that for } t \in (T, T_+), x(t) \notin C.$$

Stated briefly, eventually $x(t)$ stays out of any given compact set.
Continuation of Linear Systems

Consider the linear IVP

\[ x'(t) = A(t)x(t) + b(t), \quad x(t_0) = x_0 \quad \text{on} \quad (a, b) \quad \text{with} \quad t_0 \in (a, b), \]

where \( A(t) \in \mathbb{F}^{n\times n} \) and \( b(t) \in \mathbb{F}^n \) are continuous on \((a, b)\).

Let \( \mathcal{D} = (a, b) \times \mathbb{F}^n \). Then

\[ f(t, x) = A(t)x + b(t) \in (C, \text{Lip}_{\text{loc}}) \quad \text{on} \quad \mathcal{D}. \]

Moreover, for \( c, d \) satisfying

\[ a < c \leq t_0 \leq d < b, \]

\( f \) is uniformly Lipschitz continuous with respect to \( x \) on \([c, d] \times \mathbb{F}^n\),

\[ \text{take} \quad L = \max_{c \leq t \leq d} |A(t)|. \]

The Picard global existence theorem implies there is a solution of the IVP on \([c, d]\), which is unique by the uniqueness theorem for locally Lipschitz \( f \). This implies that \( T_- = a \) and \( T_+ = b \).