Ordinary Differential Equations (ODEs)

| · | denotes the Euclidean norm on $\mathbb{F}^n$.

|| · || denotes a function space norm.

An ODE is an equation of the form

$$g(t, x, x', \ldots, x^{(m)}) = 0,$$

where

$$g : \Omega \subset \mathbb{R} \times (\mathbb{F}^n)^{m+1} \mapsto \mathbb{F}^n.$$

A solution of this ODE on an interval $I \subset \mathbb{R}$ is a function

$$x : I \to \mathbb{F}^n$$

for which

$$x'(t), x''(t), \ldots, x^{(m)}(t) \text{ exists on } I,$$

and

$$g(t, x(t), x'(t), \ldots, x^{(m)}(t)) = 0 \quad \forall t \in I.$$

We focus on the case where $x^{(m)}$ can be solved for explicitly:

$$x^{(m)} = f(t, x, x', \ldots, x^{(m-1)}),$$

where

$$f : D \subset \mathbb{R} \times (\mathbb{F}^n)^m \mapsto \mathbb{F}^n$$

is continuous.

This equation is called an $m^{th}$-order $n \times n$ system of ODE's.

Note that if $x$ is a solution defined on an interval $I \subset \mathbb{R}$, then the existence of $x^{(m)}$ on $I$ (including one-sided limits at the endpoints of $I$) implies that $x \in C^{m-1}(I)$.

Hence $x^{(m)} \in C(I)$ since $f$ is continuous, so $x \in C^m(I)$. 
Reduction to First-Order Systems

Every $m^{th}$-order $n \times n$ system of ODE’s is equivalent to a first-order $mn \times mn$ system of ODE’s.

Define
\[ y_j(t) = x^{(j-1)}(t) \in \mathbb{F}^n \quad 1 \leq j \leq m \]
and
\[ y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \in \mathbb{F}^{mn}, \]
the system
\[ x^{(m)} = f(t, x, \ldots, x^{(m-1)}) \]
is equivalent to the first-order $mn \times mn$ system
\[ y' = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(t, y_1, \ldots, y_m) \end{bmatrix}. \]

By relabeling we can focus on first-order $n \times n$ systems of the form
\[ x' = f(t, x), \]
where
\[ f : \mathbb{R} \times \mathbb{F}^n \mapsto \mathbb{F}^n \quad \text{is continuous}. \]

Example
Consider $x'(t) = f(t)$ where $f : I \to \mathbb{F}^n$ is continuous on $I \subset \mathbb{R}$. For a fixed $t_0 \in I$, the general solution of the ODE is
\[ x(t) = c + \int_{t_0}^{t} f(s) ds, \]
where $c \in \mathbb{F}^n$ is an arbitrary.
Initial-Value Problems (IVP’s) for First-order Systems

Under certain conditions on \( f \), the general solution of a first-order system \( x' = f(t, x) \) involves \( n \) arbitrary constants in \( \mathbb{F} \).

So \( n \) scalar conditions must be given to specify a particular solution.

For the example above, clearly giving \( x(t_0) = x_0 \) determines \( c \).

An IVP for the first-order system is the differential equation
\[
DE : \quad x' = f(t, x),
\]
together with initial conditions
\[
IC : \quad x(t_0) = x_0.
\]

A solution of the IVP is a solution \( x(t) \) of the \( DE \), defined on an interval \( I \) containing \( t_0 \), which also satisfies the \( IC \).
Examples

(1) Let $n = 1$.

\[
DE: \quad x' = x^2
\]

**IVP:**

\[
\begin{align*}
IC: \quad x(1) &= 1
\end{align*}
\]

is

\[
x(t) = \frac{1}{2 - t},
\]

which blows up as $t \to 2$.

So even if $f$ is $C^\infty$ on all of $\mathbb{R} \times \mathbb{R}^n$, solutions of an IVP do not necessarily exist for all time $t$.

(2) Let $n = 1$.

\[
DE: \quad x' = 2\sqrt{|x|}
\]

**IVP:**

\[
\begin{align*}
IC: \quad x(0) &= 0.
\end{align*}
\]

For any $c \geq 0$, define

\[
x_c(t) = \begin{cases} 
0, & \text{for } t \leq c, \\
(t - c)^2, & \text{for } t \geq c.
\end{cases}
\]

Then, for every $c \geq 0$, $x_c(t)$ is a solution of this IVP.

So, in general, for continuous $f(t, x)$, IVP’s may have non-unique solutions.

The difficulty here is that $f(t, x) = 2\sqrt{|x|}$ does not satisfy a Lipschitz condition in $x$ near $x = 0$. 
An Integral Equation Equivalent to an IVP

Suppose \( x(t) \in C^1(I) \) is a solution of

\[
\begin{align*}
  \text{DE} : \quad x' &= f(t, x) \\
  \text{IC} : \quad x(t_0) &= x_0
\end{align*}
\]

(IVP)

on the interval \( I \subset \mathbb{R} \) with \( t_0 \in I \), where \( f \) is continuous.

Then for all \( t \in I \),

\[
x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds = x_0 + \int_{t_0}^{t} f(s, x(s))ds,
\]

so \( x(t) \) is also a solution of the integral equation

\[
\text{(IE)} \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \quad (t \in I).
\]

Conversely, if \( x(t) \in C(I) \) is a solution of (IE), then

\[
f(t, x(t)) \in C(I),
\]

so

\[
x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \in C^1(I)
\]

and

\[
x'(t) = f(t, x(t))
\]

by the Fundamental Theorem of Calculus.

So \( x \) is a \( C^1 \) solution of the DE on \( I \), and clearly \( x(t_0) = x_0 \), so \( x \) is a solution of (IVP).

**Proposition.** On an interval \( I \) containing \( t_0 \), \( x \) is a solution of the initial value problem (IVP) with \( x \in C^1(I) \) iff \( x \) is a solution of the integral equation (IE) on \( I \) with \( x \in C(I) \).
The Contraction Mapping Fixed-Point Theorem

The integral equation (IE) transforms the initial value problem (IVP) to a problem on $C(I)$ without concern for differentiability.

Moreover, the initial condition is built into the integral equation.

We solve (IE) using a fixed-point formulation.

**Definition.** Let $(X, d)$ be a metric space, and suppose $g : X \to X$. We say that $g$ is a *contraction* if

$$\exists c < 1 \text{ such that } d(g(x), g(y)) \leq cd(x, y) \quad \forall x, y \in X.$$  

A point $x_* \in X$ for which $g(x_*) = x_*$ is called a *fixed point* of $g$.

A contraction is a Lipschitz continuous function with Lipschitz constant $< 1$.

**The Contraction Mapping Fixed-Point Theorem**  
Let $(X, d)$ be a *complete* metric space and

$$g : X \to X$$

a contraction (with contraction constant $c < 1$).  

Then $g$ has a unique fixed point $x_* \in X$.  

Moreover, for $x_0 \in X$, if $\{x_k\}$ generated by the *fixed point iteration*

$$x_{k+1} = g(x_k) \quad \text{for} \quad k \geq 0,$$

then $x_k \to x_*$. 

Proof

Proof. Fix $x_0 \in X$, and set

$$x_{k+1} = g(x_k) \quad \text{for} \quad k \geq 0,$$

Then for $k \geq 1$,

$$d(x_{k+1}, x_k) = d(g(x_k), g(x_{k-1})) \leq cd(x_k, x_{k-1}).$$

By induction

$$d(x_{k+1}, x_k) \leq c^k d(x_1, x_0).$$

So for $n < m$,

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq \left( \sum_{j=n}^{m-1} c^j \right) d(x_1, x_0) \leq \left( \sum_{j=n}^{\infty} c^j \right) d(x_1, x_0) = \frac{c^n}{1 - c} d(x_1, x_0).$$

Since $c^n \to 0$ as $n \to \infty$, $\{x_k\}$ is Cauchy.

Since $X$ is complete, $x_k \to x_*$ for some $x_k \in X$.

Since $g$ is continuous,

$$g(x_*) = g(\lim x_k) = \lim g(x_k) = \lim x_{k+1} = x_*,$$

so $x_*$ is a fixed point.

If $x$ and $y$ are two fixed points of $g$ in $X$, then

$$d(x, y) = d(g(x), g(y)) \leq cd(x, y),$$

so

$$(1 - c)d(x, y) \leq 0.$$

Thus and $x = y$. So $g$ has a unique fixed point. $\square$
Local Existence and Uniqueness for (IVP)

Applying the Contraction Mapping Fixed-Point Theorem (C.M.F.-P.T.) to a function usually requires two steps:
(i) showing there is a complete set $S$ for which $g(S) \subset S$, and
(ii) showing that $g$ is a contraction on $S$.

To apply the C.M.F.-P.T. to the integral equation

\[ (IE) \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds, \]

we need a further condition on $f$.

**Definition.** Let $I \subset \mathbb{R}$ be an interval and $\Omega \subset \mathbb{F}^n$.
$f : I \times \Omega \mapsto \mathbb{F}^n$ is **uniformly Lipschitz continuous with respect to** $x$ if

\[ |f(t, x) - f(t, y)| \leq L|x - y| \quad (\forall t \in I)(\forall x, y \in \Omega). \]

We say that $f$ is in $(C, \text{Lip})$ on $I \times \Omega$ if $f$ is continuous on $I \times \Omega$ and $f$ is uniformly Lipschitz continuous with respect to $x$ on $I \times \Omega$.

For simplicity, we will consider intervals $I \subset \mathbb{R}$ for which $t_0$ is the left endpoint. Virtually identical arguments hold if $t_0$ is the right endpoint of $I$, or if $t_0$ is in the interior of $I$.
Theorem.

Let $\beta > 0$, $\hat{r} > 0$, and define

$$I = [t_0, t_0 + \beta] \quad \text{and} \quad \Omega = \overline{B_{\hat{r}}(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq \hat{r}\},$$

Suppose $f(t, x)$ is in $(C, \text{Lip})$ on $I \times \Omega$.

Then there exists

$$0 < \alpha \leq \beta$$

for which there is a unique solution to the integral equation

$$(IE) \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds,$$

in $C(I_\alpha)$ where

$$I_\alpha = [t_0, t_0 + \alpha].$$

Moreover, we can choose $\alpha \in (0, \beta]$ to be any positive number satisfying

$$\alpha \leq \frac{\hat{r}}{M} \quad \text{and} \quad \alpha < \frac{1}{L},$$

where

$$M = \max_{(t, x) \in I \times \Omega} |f(t, x)|$$

and $L$ is the Lipschitz constant for $f$ in $I \times \Omega$. 
Proof

For any $\alpha \in (0, \beta]$, let $\| \cdot \|_\infty$ denote the max-norm on $C(I_\alpha)$ (i.e. the uniform convergence norm).
Then $(C(I_\alpha), \| \cdot \|_\infty)$ is a Banach space.

Let $\tilde{x}_0$ denote the constant function $\tilde{x}_0(t) \equiv x_0$ in $C(I_\alpha)$.

Define

$$X_{\alpha,r} = \{ x \in C(I_\alpha) : \| x - \tilde{x}_0 \|_\infty \leq r \}.$$

Then $X_{\alpha,r}$ is a complete metric space since it is a closed subset of the Banach space $(C(I_\alpha), \| \cdot \|_\infty)$.

Define $g : X_{\alpha,r} \to C(I_\alpha)$ by

$$(g(x))(t) = x_0 + \int_{t_0}^t f(s, x(s))ds .$$

The mapping $g$ is well-defined on $X_{\alpha,r}$ and $g(x) \in C(I_\alpha)$ for $x \in X_{\alpha,r}$ since $f$ is continuous on $I \times \overline{B_r(x_0)}$.

Fixed points of $g$ are solutions of the integral equation (IE).
Claim Suppose $\alpha \in (0, \beta]$ and 
\[ \alpha \leq \min \left\{ \frac{r}{M}, \frac{1}{L} \right\}. \]

Then $g$ maps $X_{\alpha, r}$ into itself and $g$ is a contraction on $X_{\alpha, r}$ with contraction coefficient $\alpha L$.

Proof If $x \in X_{\alpha, r}$, then for $t \in I_{\alpha}$,
\[ |(g(x))(t) - x_0| \leq \int_{t_0}^{t} |f(s, x(s))| ds \leq M\alpha \leq r, \]
so $g : X_{\alpha, r} \rightarrow X_{\alpha, r}$.

If $x, y \in X_{\alpha, r}$, then for $t \in I_{\alpha}$,
\[ |(g(x))(t) - (g(y))(t)| \leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| ds \]
\[ \leq \int_{t_0}^{t} L|x(s) - y(s)| ds \]
\[ \leq \leq L\alpha \|x - y\|_{\infty}, \]
so
\[ \|g(x) - g(y)\|_{\infty} \leq L\alpha \|x - y\|_{\infty}, \]
where $L\alpha < 1$, that is $g$ is a contraction on $X_{\alpha, r}$. \qed
By the C.M.F.-P.T., \( g \) has a unique fixed point in \( X_{\alpha,r} \). Thus the integral equation (IE) has a unique solution \( x_*(t) \) in

\[
X_{\alpha,r} = \{ x \in C(I_\alpha) : ||x - \tilde{x}_0||_\infty \leq r \}.
\]

We now show uniqueness.

Fix \( \alpha > 0 \).

For \( 0 < \gamma \leq \alpha \), \( x_*|_{I_\gamma} \) is the unique fixed point of \( g \) on \( X_{\gamma,r} \).

Suppose \( y \in C(I_\alpha) \) is a solution of (IE) on \( I_\alpha \) with \( y \not\equiv x_\alpha \) on \( I_\alpha \). Let

\[
\gamma_1 = \inf\{ \gamma \in (0, \alpha] : y(t_0 + \gamma) \neq x_*(t_0 + \gamma) \}.
\]

By continuity, \( \gamma_1 < \alpha \).

Since \( y(t_0) = x_0 \), continuity implies

\[
\exists \gamma_0 \in (0, \alpha] \text{ such that } y|_{I_{\gamma_0}} \in X_{\gamma_0,r}.
\]

Thus \( y(t) \equiv x_*(t) \) on \( I_{\gamma_0} \).

So \( 0 < \gamma_1 < \alpha \). Since \( y(t) \equiv x_*(t) \) on \( I_{\gamma_1} \),

\[
y \bigg|_{I_{\gamma_1}} \in X_{\gamma_1,r}.
\]

Let \( \rho = M\gamma_1 \); then \( \rho < M\alpha \leq r \). For \( t \in I_{\gamma_1} \),

\[
|y(t) - x_0| = |(g(y))(t) - x_0| \leq \int_{0}^{t} |f(s,y(s))|ds \leq M\gamma_1 = \rho,
\]

so \( y \bigg|_{I_{\gamma_1}} \in X_{\gamma_1,\rho} \).

By continuity,

\[
\exists \gamma_2 \in (\gamma_1, \alpha] \text{ such that } y \bigg|_{I_{\gamma_2}} \in X_{\gamma_2,r}.
\]

But then \( y(t) \equiv x_*(t) \) on \( I_{\gamma_2} \), contradicting the definition of \( \gamma_1 \). \( \square \)