The Resolvent

Let $V$ be a finite-dimensional vector space and $L \in \mathcal{L}(V)$.

If $\zeta \not\in \sigma(L)$, then the operator $L - \zeta I$ is invertible.

Define

$$R(\zeta) = (L - \zeta I)^{-1}$$

(sometimes denoted $R(\zeta, L)$).

The function $R : \mathbb{C} \setminus \sigma(L) \rightarrow \mathcal{L}(V)$ is called the resolvent of $L$.

$R(\zeta)$ provides an analytic approach to questions about the spectral theory of $L$.

The set $\mathbb{C} \setminus \sigma(L)$ is called the resolvent set of $L$.

Since the inverses of commuting invertible linear transformations also commute,

$$R(\zeta_1) \text{ and } R(\zeta_2) \text{ commute } \forall \ \zeta_1, \zeta_2 \in \mathbb{C} \setminus \sigma(L).$$

Since a linear transformation commutes with its inverse, it also follows that $L$ commutes with all $R(\zeta)$. 
Basic Resolvent Equations

Let $\zeta_1, \zeta_2 \in \mathbb{C} \setminus \sigma(L)$.

\[ R(\zeta_1) - R(\zeta_2) = R(\zeta_1)(L - \zeta_2)R(\zeta_2) - R(\zeta_1)(L - \zeta_1)R(\zeta_2) \]
\[ = R(\zeta_1)((L - \zeta_2) - (L - \zeta_1))R(\zeta_2) \]
\[ = (\zeta_1 - \zeta_2)R(\zeta_1)R(\zeta_2) \]

\[ R(\zeta_1)R(\zeta_2) = (\zeta_1 - \zeta_2)^{-1}[R(\zeta_1) - R(\zeta_2)] \]

\[ R(\zeta_1) = [I - (\zeta_2 - \zeta_1)R(\zeta_1)]R(\zeta_2) \]

\[ R(\zeta_2) = [I - (\zeta_2 - \zeta_1)R(\zeta_1)]^{-1}R(\zeta_1) \]
$R(\zeta)$ is Holomorphic on $\mathbb{C}\setminus \sigma(L)$

Let $\zeta, \zeta_0 \in \mathbb{C}\setminus \sigma(L)$.

Apply results for Neumann series:

$$R(\zeta) = [1 - (\zeta - \zeta_0)R(\zeta_0)]^{-1}R(\zeta_0)$$

$$= \sum_{n=0}^{\infty} (\zeta - \zeta_0)^n R(\zeta_0)^{n+1}$$

with this series being absolutely convergent if

$$|\zeta - \zeta_0| < \|R(\zeta_0)\|^{-1}.$$  

This is just the Taylor series expansion of $R$ at $\zeta = \zeta_0$.

Hence $R$ is holomorphic on $\mathbb{C}\setminus \sigma(L)$ with

$$R^{(n)}(\zeta) = n!R(\zeta)^{n+1} \quad n = 1, 2, \ldots.$$  

In addition, for $|\zeta|$ large, $|\zeta|^{-1}\|L\| < 1$,

$$R(\zeta) = -\zeta^{-1}(1 - \zeta^{-1}L)^{-1}$$

$$= -\sum_{n=0}^{\infty} \zeta^{-(n+1)}L^n$$

which is absolutely convergent if $|\zeta| > \|L\|$.

Thus, $R(\zeta)$ is holomorphic at $\infty$, and $R(\zeta) \to 0$ as $\zeta \to \infty$.  

Cauchy Integral Formulas and Laurent Series

*Cauchy Integral Formulas*
If \( f \) is analytic inside and on a simple closed curve \( C \) and \( z \) is any point inside \( C \), then

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta.
\]

Moreover, the \( n^{\text{th}} \) derivative of \( f \) at \( z \) is given by

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n = 0, 1, 2, \ldots.
\]

*Laurent Series Expansions*
If \( f \) is analytic inside and on the boundary of the annular shaped region \( R \) bounded by two concentric circles \( C_1 \) and \( C_2 \) with center at \( z_0 \) and respective radii \( r_1 \) and \( r_2 \) (\( r_1 > r_2 \)), then for all \( z \) in \( R \),

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n
\]

where

\[
a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \quad n = 0, 1, 2, \ldots
\]

\[
a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{(\zeta - a)^{-n+1}} d\zeta, \quad n = 1, 2, \ldots
\]

The first part of the Laurent expansion is called the *principal part* and the second is called the *analytic part*. 
Singularities and Residue

Poles:
If \( f(z) \) has Laurent expansion in which the principle part has only finitely many terms given by

\[
\frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-n}}{(z - z_0)^n}
\]

where \( a_{-n} \neq 0 \), then \( z_0 \) is a pole of order \( n \) for \( f \).
If \( n = 1 \), then it is a simple pole.

Essential Singularities:
If \( f(z) \) has Laurent expansion in which the principle part has infinitely many terms, then \( z_0 \) is an essential singularity for \( f \).

A function is entire if it is analytic on all of \( \mathbb{C} \).

If \( f \) is analytic on a region \( \Omega \) except for finitely many poles, then \( f \) is said to be meromorphic on \( \Omega \).

In the Laurent expansion of \( f \) the coefficient \( a_{-1} \) is called the residue of \( f \).

The Residue Theorem
Let \( f \) be single-valued and analytic inside and on a simple closed curve \( C \) except at the

singularities \( z_1, \ldots, z_n \)

inside \( C \) having

residues \( a_{-1}^1, a_{-1}^2, \ldots, a_{-1}^n \).

Then

\[
\oint_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} a_{-1}^k.
\]
The Laurent Series of the Resolvent

The resolvent is meromorphic with poles at each eigenvalue.

For the sake of simplicity we assume that $0 \in \sigma(L)$ and we compute the Laurent series at the origin.

$$R(\zeta) = \sum_{n=-\infty}^{\infty} \zeta^n A_n,$$

where

$$A_n = \frac{1}{2\pi i} \oint_{\Gamma} \zeta^{-(n+1)} R(\zeta) d\zeta \quad \forall \ n.$$ 

Let $\tilde{\Gamma}$ be a contour around the origin slightly larger than $\Gamma$, then

$$A_n A_m = \left( \frac{1}{2\pi i} \right)^2 \oint_{\Gamma} \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} \omega^{-(m+1)} R(\omega) R(\zeta) d\omega d\zeta$$

$$= \left( \frac{1}{2\pi i} \right)^2 \oint_{\Gamma} \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} [R(\omega) - R(\zeta)] d\omega d\zeta.$$

By the Residue Theorem and geometric series,

$$\eta_n \omega^{-(n+1)} = \frac{1}{2\pi i} \oint_{\Gamma} \zeta^{-(n+1)} (\omega - \zeta)^{-1} d\zeta$$

$$\quad (1 - \eta_m) \zeta^{-(m+1)} = \frac{1}{2\pi i} \oint_{\Gamma} \omega^{-(m+1)} (\omega - \zeta)^{-1} d\omega,$$

with

$$\eta_n = \begin{cases} 
 1, & \text{for } n \geq 0 \\
 0, & \text{else.}
\end{cases}$$
Therefore,

\[ A_n A_m = \left( \frac{1}{2\pi i} \right)^2 \oint_\Gamma \oint_\Gamma \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} [R(\omega) - R(\zeta)] d\omega d\zeta \]

\[ = \left( \frac{1}{2\pi i} \right)^2 \oint_\Gamma \oint_\Gamma \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} R(\omega) d\omega d\zeta \]

\[ - \left( \frac{1}{2\pi i} \right)^2 \oint_\Gamma \oint_\Gamma \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} R(\zeta) d\omega d\zeta \]

\[ = \left( \frac{1}{2\pi i} \right)^2 \oint_\Gamma \omega^{-(m+1)} R(\omega) \oint_\Gamma \zeta^{-(n+1)} (\omega - \zeta)^{-1} d\zeta d\omega \]

\[ - \left( \frac{1}{2\pi i} \right)^2 \oint_\Gamma \zeta^{-(n+1)} R(\zeta) \oint_\Gamma \omega^{-(m+1)} (\omega - \zeta)^{-1} d\omega d\zeta \]

\[ = \eta_n \frac{1}{2\pi i} \oint_\Gamma \omega^{-(m+n+1) + 1} R(\omega) d\omega \]

\[ + (\eta_m - 1) \frac{1}{2\pi i} \oint_\Gamma \zeta^{-(m+n+1) + 1} R(\zeta) d\zeta \]

\[ = (\eta_m + \eta_n - 1) A_{m+n+1} . \]
\[ \xi_n = \begin{cases} 1, & n \geq 0, \\ 0, & \text{else.} \end{cases} \]

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>n+m+1</th>
<th>( A_n A_m = (\xi_n + \xi_m - 1) A_{n+m+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>( A_{-1}^2 = -A_{-1} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(-A_{-1} = P) a projection</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( N = -A_2 )</td>
</tr>
<tr>
<td>-2</td>
<td>-2</td>
<td>-3</td>
<td>( A_{-2}^2 = -A_{-3} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(-A_{-3} = N^2 )</td>
</tr>
<tr>
<td>-2</td>
<td>-3</td>
<td>-4</td>
<td>( A_{-2} A_{-3} = -A_{-4} )</td>
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<td></td>
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<td>(-A_{-4} = N^3 )</td>
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<tr>
<td>-2</td>
<td>-4</td>
<td>-5</td>
<td>( A_{-2} A_{-4} = -A_{-5} )</td>
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<td></td>
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<td></td>
<td>(-A_{-5} = N^4 )</td>
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<td>(-A_{-k} = N^{k-1} )</td>
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<td>( S = A_0 )</td>
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<td>0</td>
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<td>( A_0^2 = A_1 )</td>
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<td>( A_1 = S^2 )</td>
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<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>( A_0 A_1 = A_2 )</td>
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<td>( A_2 = S^3 )</td>
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<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>( A_0 A_2 = A_3 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( A_3 = S^4 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( A_n = S^{n+1} \ n \geq 0 )</td>
</tr>
</tbody>
</table>
Therefore, for $\lambda_j \in \sigma(T)$ \(\exists P_j, N_j, S_j\) such that

\[
R(\zeta) = \left[ - (\zeta - \lambda_j)^{-1} P_j - \sum_{n=1}^{\infty} (\zeta - \lambda_j)^{-n} N_j^n \right] + \left[ \sum_{n=0}^{\infty} (\zeta - \lambda_j)^n S_j^{(n+1)} \right] = C_j(\zeta) + S_j(\zeta).
\]

Moreover, the table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$n+m+1$</th>
<th>$A_n A_m = (\xi_n + \xi_m - 1) A_{n+m+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-2</td>
<td>-2</td>
<td>$N_j = -A_{-2} = A_{-1}A_{-2} = P_j N_j$</td>
</tr>
<tr>
<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>$N_j = -A_{-2} = A_{-2}A_{-1} = N_j P_j$</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>$0 = 0 \cdot A_0 = A_{-1}A_0 = -P_j S_j$</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>$0 = 0 \cdot A_0 = A_0 A_{-2} = -S_j P_j$</td>
</tr>
</tbody>
</table>

Hence the decomposition

\[
C_j(\zeta) + S_j(\zeta)
\]

is a direct sum decomposition of $R$ compatible with

\[
V = M_j \oplus M'_j,
\]

where

\[
M_j = P_j V \quad \text{and} \quad M'_j = (I - P_j) V.
\]
The principal part of the Laurent expansion for $R(\zeta)$ at $\lambda_j \in \sigma(L)$ acts on the subspace $M_j$.

In particular, $R(\zeta)$ has an isolated singularity at $\zeta = \lambda_j$ but is otherwise convergent.

Thus, in particular,

$$\sum_{h=1}^{\infty} (\zeta - \lambda_j)^{-(n+1)} N_j^n$$

is absolutely convergent on $\mathbb{C} \setminus \{\lambda_j\}$.

Setting $\zeta = \lambda_j + \xi$ for $\xi \neq 0$, we obtain

$$\sum_{n=1}^{\infty} \xi^{-(n+1)} N_j^n < \infty.$$ 

Therefore, $\rho(N) \leq |\xi|$ for all $\xi \in \mathbb{C}$.

Consequently, $\rho(N) = 0$ and so $N_j$ is nilpotent with

$$\text{rank} \,(N_j) < \text{rank} \,(P_j)$$

where

$$\text{rank} \,(P_j) = \dim(M_j) = m_j.$$ 

Hence, $\lambda_j$ is a pole of $R(\zeta)$ of order less than or equal to $\text{rank} \,(P_j)$ since the principal part the Laurent series at $\lambda_j$ is finite.

Therefore, $R(\zeta)$ is meromorphic!
If $\sigma(T) = \{\lambda_1, \ldots, \lambda_s\}$, then

\begin{align*}
(2) \quad P_j P_k &= \delta_{jk} P_j \\
(3) \quad \sum_{j=1}^s P_j &= I \\
(4) \quad P_j L &= L P_j.
\end{align*}

Proof of (1):

\[ P_j P_k = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_j} \int_{\Gamma_k} R(\zeta) R(w) dw d\zeta, \]

where the regions defined by $\Gamma_j + \Gamma_k$ do not overlap.

\begin{align*}
P_j P_k &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_j} \int_{\Gamma_k} (w - \zeta)^{-1} [R(w) - R(\zeta)] dw d\zeta \\
&= \left(\frac{1}{2\pi i}\right)^2 \left[ \int_{\Gamma_k} \left[ \int_{\Gamma_j} (w - \zeta)^{-1} R(w) d\zeta \right] dw \\
&\quad - \int_{\Gamma_j} \left[ \int_{\Gamma_k} (w - \zeta)^{-1} R(\zeta) dw \right] d\zeta \right] \\
&= 0 \quad j \neq k.
\end{align*}
Proof of (2):
Let $\Gamma$ be a simple closed curve containing all the singularities of $R(\zeta)$. Then

$$\frac{1}{2\pi i} \int_\Gamma R(\zeta) \, ds = \text{sum of the residues} = -\sum_{j=1}^{s} P_j.$$ 

Also, from the expansion of $R(\zeta)$ at $\infty$, we have

$$\frac{1}{2\pi i} \int_\Gamma R(\zeta) \, ds = \frac{1}{2\pi i} \int_\Gamma -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n \, ds$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} L^n \left( \int_\Gamma \zeta^{-(n+1)} \, d\zeta \right)$$

$$= -I.$$

Proof of (3):
$L$ commutes with $R(\zeta)$ so that $L$ commutes with

$$P_j = \frac{1}{2\pi i} \int_{\Gamma_j} R(\zeta) \, d\zeta.$$
The Resolvent

Let $V$ be a finite-dimensional vector space and $L \in \mathcal{L}(V)$. If $\zeta \not\in \sigma(L)$, then the operator $L - \zeta I$ is invertible.

Define the resolvent of $L$ as

$$R(\zeta) = (L - \zeta I)^{-1}.$$ 

We have shown that for each $\lambda_j \in \sigma(T)$, there exist $P_j, N_j, S_j$ such that

$$R(\zeta) = \left[-(\zeta - \lambda_j)^{-1}P_j - \sum_{n=1}^{m_j}(\zeta - \lambda_j)^{-(n+1)}N_j^n\right]$$

$$+ \left[\sum_{n=0}^{\infty}(\zeta - \lambda_j)^nS_j^{(n+1)}\right]$$

$$= C_j(\zeta) + S_j(\zeta) ,$$

where $m_j = \text{rank}(P_j)$. 

$$N_jP_j = N_j = P_jN_j \quad P_jS_j = 0 = S_jP_j$$

$$P_jP_k = \delta_{jk}P_j$$

$$\sum_{j=1}^{s}P_j = I$$

$$P_jL = LP_j.$$
Partial Fractions Decomposition of $R(\zeta)$

Let $C_j(\zeta)$ be the principal part of $R(\zeta)$ at each of its poles

$$\sigma(L) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}.$$ 

Then the function

$$F(\zeta) = R(\zeta) - \sum_{j=1}^{s} C_j(\zeta)$$

has an analytic extension to all of $\mathbb{C}$ since the poles $\lambda_j$ are removable. Moreover,

$$\lim_{\zeta \to \infty} F(\zeta) = 0$$

since

$$\lim_{\zeta \to \infty} C_j(\zeta) = 0 \quad \text{and} \quad \lim_{\zeta \to \infty} R(\zeta) = 0 :$$

$$R(\zeta) = -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n .$$

Therefore, $F(\zeta)$ is a bounded entire function. By Liouville’s Theorem $F \equiv 0$.

Hence,

$$R(\zeta) = \sum_{j=1}^{s} C_j(\zeta)$$

$$= -\sum_{j=1}^{s} \left[ (\zeta - \lambda_j)^{-1} P_j + \sum_{n=1}^{m_j-1} (\zeta - \lambda_j)^{-(n+1)} N_j \right] .$$
Spectral Representation Theorem

Since $X = M_1 \oplus \cdots \oplus M_s$, we can write $L$ as the sum of the operators $P_j L P_j, j = 1, \ldots, s$.

Note that for each $j = 1, \ldots, s$

$$P_j L P_j = L P_j = \lambda_j P_j + N_j.$$

Therefore,

$$L = S + N$$

where

$$S = \sum_{j=1}^{s} \lambda_j P_j \quad \text{and} \quad N = \sum_{j=1}^{s} N_j,$$

with

$$I = \sum_{j=1}^{s} P_j$$

$$P_k P_j = P_j P_k = \delta_{jk} P_j$$

$$N_j P_k = P_k N_j = \delta_{kj} N_j$$

$$N_j P_k = P_k N_j = 0 \quad i \neq j.$$

This representation is unique in the sense that any other such representation $L = S' + N'$ has $S = S'$ and $N = N'$.
Some Terminology

\( M_j \sim \hat{E}_j(L) \) the algebraic eigenspace for \( \lambda_j \)

\( \text{dim}(M_j) \sim \text{algebraic multiplicity of } \lambda_j. \)

\( P_j \sim \text{eigenprojection for the eigenvalue } \lambda_i. \)

\( N_j \sim \text{e-nilpotent for the eigenvalue } \lambda_j. \)

\( u \in M_j \backslash \{0\} \) — generalized eigenvectors of \( \lambda_j \)

\( \lambda_j \in \sigma(L) \)

\( \lambda_j \) is said to be simple if \( m_j = 1 \) (\( N_j = 0 \)).

\( \lambda_j \) is said to be semi-simple if \( N_j = 0 \).

\( \lambda_j \) is said to be degenerate if \( \lambda_j \) is not simple.

\( \lambda_j \) is said to be non-derogatory if \( \text{rank}(N_j) = m_j \).

\( \lambda_j \) is said to be derogatory if \( \text{rank}(N_j) < m_j \).
The Cauchy Integral Formula (CIF) for Operators

Let
\[ \phi(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots \]
be a power series with radius of convergence \( r > 0 \).

Then \( \phi \) is holomorphic on the disc \( |\zeta| < r \).

If \( L \in \mathcal{B}(V) \) has \( \|L\| < r \), then as we have seen the operator
\[ \phi(L) = a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \ldots \]
is well defined with the series being absolutely convergent.

**Theorem**
The mapping \( \phi : \{ T \in \mathcal{B}(V) : \|T\| < r \} \rightarrow \mathcal{B}(V) \) defined above
satisfies the Cauchy integral formula. That is
\[
\phi(L) = -\frac{1}{2\pi i} \oint_{\Gamma} \phi(\zeta) R(\zeta, L) d\zeta = \frac{1}{2\pi i} \oint_{\Gamma} \phi(\zeta)(\zeta - L)^{-1} d\zeta,
\]
where \( \Gamma \) is any simple closed curve contained within the disc of radius \( r \).
Proof

Recall that $\rho(L) \leq ||L||$, so for $||L|| < |\zeta|$, 

$$R(\zeta) = -\zeta^{-1} (1 - \zeta^{-1} L)^{-1}$$

$$= - \sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n .$$

Hence,

$- \phi(\zeta) R(\zeta)$

$$= (a_0 + a_1 \zeta + a_2 \zeta^2 + \ldots) \left( \frac{1}{\zeta} + \frac{L}{\zeta^2} + \frac{L^2}{\zeta^3} + \ldots \right)$$

$$= \frac{1}{\zeta} \left( a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \ldots \right) + \left( a_1 + a_2 L + a_3 L^2 + \ldots \right)$$

$$+ \frac{L}{\zeta^2} \left( a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \ldots \right) + \zeta \left( a_2 + a_3 L + a_4 L^2 + \ldots \right)$$

$$+ \frac{L^2}{\zeta^2} \left( a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \ldots \right) + \zeta^2 \left( a_3 + a_4 L + a_5 L^2 + \ldots \right)$$

$$\vdots$$

Hence, by the Residue Theorem (and uniform convergence),

$$- \frac{1}{2\pi i} \int_{\Gamma} \phi(\zeta) R(\zeta, L) d\zeta = \left( a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \ldots \right)$$

$$= \phi(L) .$$
Operator Algebras of Holomorphic Functions

**Proposition.** Suppose $L \in \mathcal{L}(V)$ and $\varphi_1$ and $\varphi_2$ are both holomorphic in a neighborhood of $\sigma(L)$. Then

(a) $(a_1 \varphi_1 + a_2 \varphi_2)(L) = a_1 \varphi_1(L) + a_2 \varphi_2(L)$, and

(b) $(\varphi_1 \varphi_2)(L) = \varphi_1(L) \circ \varphi_2(L)$.

**Proof**
(a) follows from the linearity of contour integration.

To see (b) let $\Omega$ be the domain on which both $\varphi_1$ and $\varphi_2$ are holomorphic and which contains $\sigma(L)$. Let

$$\lambda_1, \ldots, \lambda_k$$

be the distinct eigenvalues of $L$, with algebraic multiplicities

$$m_1, \ldots, m_k,$$

respectively.

For $s = 1, 2$, $j = 1, 2, \ldots, k$, let $\Gamma_{sj}$, $s = 1, 2$, be two circles around $\lambda_j$ with the radius of $\Gamma_{2j}$ greater than that of $\Gamma_{1j}$ and such that the discs $\Delta_{sj}$, $s = 1, 2$, associated with $\Gamma_{sj}$, resp. ly, are contained in $\Omega$.

Set $\Gamma_s = \bigcup_{j=1}^k \Gamma_{sj}$, $s = 1, 2$. 
By the first resolvent equation we get
\[ \varphi_1(L) \circ \varphi_2(L) \]
\[ = \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \varphi_1(\zeta) R(\zeta_1) d\zeta_1 \circ \oint_{\Gamma_2} \varphi_2(\zeta_2) R(\zeta_2) d\zeta_2 \]
\[ = \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \varphi(\zeta_1) \varphi(\zeta_2) R(\zeta_1) \circ R(\zeta_2) d\zeta_2 d\zeta_1 \]
\[ = \frac{1}{(2\pi i)^2} \left[ \oint_{\Gamma_1} \varphi_1(\zeta_1) R(\zeta_1) \oint_{\Gamma_2} \frac{\varphi_2(\zeta_2)}{\zeta_1 - \zeta_2} d\zeta_2 d\zeta_1 \right. \]
\[ \left. - \oint_{\Gamma_2} \varphi_2(\zeta_2) R(\zeta_2) \oint_{\Gamma_1} \frac{\varphi_1(\zeta_1)}{\zeta_1 - \zeta_2} d\zeta_1 d\zeta_2 \right] \]

Since \( \zeta_1 \) is inside \( \Gamma_2 \) and \( \zeta_2 \) is outside \( \Gamma_1 \), the CIF gives
\[ 0 = \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{\varphi_1(\zeta_1)}{\zeta_1 - \zeta_2} d\zeta_1, \quad \text{and} \]
\[ \varphi_2(\zeta_1) = \frac{1}{2\pi i} \oint_{\Gamma_2} \frac{\varphi_2(\zeta_2)}{\zeta_2 - \zeta_1} d\zeta_2. \]

Therefore,
\[ \varphi_1(L) \circ \varphi_2(L) = -\frac{1}{2\pi i} \oint_{\Gamma_1} \varphi_1(\zeta_1) \varphi_2(\zeta_1) R(\zeta_1) d\zeta_1 = (\varphi_1 \varphi_2)(L). \]
Remarks

(1) Since

$$\varphi_1 \varphi_2)(\zeta) = (\varphi_2 \varphi_1)(\zeta),$$

(b) implies that \(\varphi_1(L)\) and \(\varphi_2(L)\) always commute.

(2) Suppose \(L \in \mathcal{L}(V)\) is invertible and \(\varphi(\zeta) = \frac{1}{\zeta}\). Since \(\sigma(L) \subset \mathbb{C}\setminus\{0\}\) and \(\varphi\) is holomorphic on \(\mathbb{C}\setminus\{0\}\), \(\varphi(L)\) is defined.
Since \(\zeta \cdot \frac{1}{\zeta} = \frac{1}{\zeta} \cdot \zeta = 1\), \(L\varphi(L) = \varphi(L)L = I\). Thus \(\varphi(L) = L^{-1}\), as expected.

(3) One can show similarly that if

$$\varphi(\zeta) = \frac{p(\zeta)}{q(\zeta)}$$

is a rational function, i.e.\(p, q\) are polynomials, and

$$\sigma(L) \subset \{\zeta : q(\zeta) \neq 0\},$$

then

$$\varphi(L) = p(L)q(L)^{-1},$$

as expected.
The Spectral Mapping Theorem

Suppose $L \in \mathcal{L}(V)$ and $\varphi$ is holomorphic in a neighborhood of $\sigma(L)$ (so $\varphi(L)$ is well-defined).
Then
$$
\sigma(\varphi(L)) = \varphi(\sigma(L))
$$
including multiplicities, i.e., if
$$
\mu_1, \ldots, \mu_n
$$
are the eigenvalues of $L$ counting multiplicities, then
$$
\varphi(\mu_1), \ldots, \varphi(\mu_n)
$$
are the eigenvalues of $\varphi(L)$ counting multiplicities.

Proof
Let $\Omega$ be the domain on which $\varphi$ is holomorphic and let
$$
\lambda_1, \ldots, \lambda_k
$$
be the distinct eigenvalues of $L$, with algebraic multiplicities
$$
m_1, \ldots, m_k,
$$
respectively.

Let $\Gamma$ be the union of $k$ simple closed curves $\Gamma_j$, where each $\Gamma_j$ is a circle around $\lambda_j$ and such that the disc $\Delta_j$ associated with $\Gamma_j$ is contained in $\Omega$. 
By the residue theorem,
\[ \varphi(L) = -\frac{1}{2\pi i} \oint_{\Gamma} \varphi(\zeta) R(\zeta) d\zeta \]

\[ = -\sum_{i=1}^{k} \text{Res}_{\zeta=\lambda_i} [\varphi(\zeta) R(\zeta)]. \]

By the partial fractions decomposition of the resolvent,
\[ -R(\zeta) = \sum_{i=1}^{k} \left( \frac{P_i}{\zeta - \lambda_i} + \sum_{\ell=1}^{m_i-1} (\zeta - \lambda_i)^{-1} N_i^\ell \right). \]

It follows that
\[ -\text{Res}_{\zeta=\lambda_i} \varphi(\zeta) R(\zeta) = \varphi(\lambda_i) P_i + \sum_{\ell=1}^{m_i-1} \text{Res}_{\zeta=\lambda_i} [\varphi(\zeta)(\zeta - \lambda_i)^{-1} N_i^\ell] \]

\[ = \varphi(\lambda_i) P_i + \sum_{\ell=1}^{m_i-1} \frac{1}{\ell!} \varphi^{(\ell)}(\lambda_i) N_i^\ell. \]

Thus
\[ (*) \quad \varphi(L) = \sum_{i=1}^{k} [\varphi(\lambda_i) P_i + \sum_{\ell=1}^{m_i-1} \frac{1}{\ell!} \varphi^{(\ell)}(\lambda_i) N_i^\ell] \]

By the uniqueness of the spectral decomposition of an operator, this must be the explicit formula for the spectral decomposition of \( \varphi(L) \)!
Spectral Properties of Composition

Proposition. Let $L \in \mathcal{L}(V)$.

Suppose $\varphi_1$ is holomorphic in a neighborhood of $\sigma(L)$, and

$\varphi_2$ is holomorphic in a neighborhood of

$$\sigma(\varphi_1(L)) = \varphi_1(\sigma(L)).$$

So $\varphi_2 \circ \varphi_1$ is holomorphic in a neighborhood of $\sigma(L)$.

Then

$$(\varphi_2 \circ \varphi_1)(L) = \varphi_2(\varphi_1(L)).$$

Proof

Let $\Delta_2$ a the union of discs containing $\sigma(\varphi_1(L))$ with $\varphi_2$ holomorphic on $\Delta_2$, and let $\Gamma_2 = \partial \Delta_2$.

Then

$$\varphi_2(\varphi_1(L)) = \frac{1}{2\pi i} \oint_{\Gamma_2} \varphi_2(\zeta_2)(\zeta_2 - \varphi_1(L))^{-1} d\zeta_2.$$

For each fixed $\zeta_2 \in \Gamma_2$, consider the function

$$f(\zeta_1) = (\zeta_2 - \varphi_1(\zeta_1))^{-1}.$$

Let $\Delta_1$ be a union of discs containing $\sigma(L)$ and chosen so small that

$\varphi_1$ holomorphic on $\Delta_1$ and

$$\varphi_1(\Delta_1) \subset \text{int}(\Delta_2).$$

Set $\Gamma_1 = \partial \Delta_1$.

Then at each point of $\Gamma_2$, the function $f$ is holomorphic on an open set containing $\Delta_1$.

Therefore, by the operator version of the CFI,

$$f(L) = (\zeta_2 - \varphi_1(L))^{-1} = -\frac{1}{2\pi i} \oint_{\Gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1.$$
Plugging the expression

\[ (\zeta_2 - \varphi_1(L))^{-1} = -\frac{1}{2\pi i} \oint_{\Gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1 \]

into

\[ \varphi_2(\varphi_1(L)) = \frac{1}{2\pi i} \oint_{\Gamma_2} \varphi_2(\zeta_2)(\zeta_2 - \varphi_1(L))^{-1} d\zeta_2 \]

gives

\[ \varphi_2(\varphi_1(L)) = -\frac{1}{(2\pi i)^2} \oint_{\Gamma_2} \varphi_2(\zeta_2) \oint_{\Gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1 d\zeta_2 \]

\[ = -\frac{1}{(2\pi i)^2} \oint_{\Gamma_1} R(\zeta_1) \oint_{\Gamma_2} \frac{\varphi_2(\zeta_2)}{\zeta_2 - \varphi_1(\zeta_1)} d\zeta_2 d\zeta_1 \]

\[ = -\frac{1}{2\pi i} \oint_{\Gamma_1} R(\zeta_1) \varphi_2(\varphi_1(\zeta_1)) d\zeta_1 \]

\[ = (\varphi_2 \circ \varphi_1)(L). \]
Logarithms of Invertible Matrices

As an application of the theory given above, we consider logarithms of invertible matrices.

Let \( L \in \mathbb{L}(V) \) be invertible.

We could define the logarithm using power series. That is, one could define

\[
\log(I + L) = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{L^\ell}{\ell}.
\]

But this series only converges absolutely in norm for a restricted class of \( L \), namely \( \{ A : \rho(A) < 1 \} \).

Let us now take an operator approach.

Choose a branch of \( \log \zeta \) holomorphic in a neighborhood of \( \sigma(L) \).

Next choose an appropriate region \( \Omega \) in which \( \log \zeta \) is defined.

In this context, by *region*, we mean that \( \Omega \) is open and \( \Gamma = \partial \Omega \) is a simple closed curve.

Form

\[
\log L = -\frac{1}{2\pi i} \int_{\Gamma} \log \zeta R(\zeta) d\zeta.
\]

This definition depends on the particular branch chosen, but since \( e^{\log \zeta} = \zeta \) for any such branch, it follows that for any such choice, \( e^{\log L} = L \).

Hence, *every* invertible matrix is in the range of the exponential! This is much better than one can do with series.