The Hanh-Banach Theorem (Geometric Form)

**Proposition.** If \((V, \| \cdot \|)\) is a normed linear space and \(v \in V\), then there exists \(f \in V^*\) such that
\[
f(v) = \|v\| \|f\|.
\]

**Proposition.** *(Geometric Form of the Hanh-Banach Theorem)*
Let \(C\) be a nonempty convex subset of the normed linear space \((V, \| \cdot \|)\). If \(v \in V\) is such that \(v\) is not in \(C\), then there exists \(f \in V^*\) such that
\[
\Re f(v) \geq \sup_{u \in C} \Re f(u).
\]

The Geometric Form of the Hahn-Banach Theorem is also known as the *fundamental separation theorem for convex sets* on a normed linear space. The linear functional \(f\) in the Proposition is said to *separate* \(v\) from the set \(C\). Equivalently, \(f\) determines a a hyperplane such that \(v\) and \(C\) lie in opposing closed half spaces.
Consequences of the Hahn-Banach Theorem

The Second Dual

\((V, \| \cdot \|)\) a normed linear space.
\((V^*, \| \cdot \|)\) the dual normed linear space.

Given \(v \in V\), define \(v^{**} \in V^{**}\) by
\[ v^{**}(f) = f(v) . \]

Then
\[ |v^{**}(f)| = |f(v)| \leq \|f\| \cdot \|v\| \implies \|v^{**}\| \leq \|v\| \]

The Hahn-Banach theorem \(\Rightarrow\)
\[ \exists f \in V^* \ni \|f\| = 1 \text{ and } v^{**}(f) = f(v) = \|v\| , \]
so
\[ \|v^{**}\| = \sup_{\|f\|=1} |v^{**}(f)| \geq \|v\| . \]

Hence \(\|v^{**}\| = \|v\|\).

Therefore, the mapping \(v \mapsto v^{**}\) from \(V\) into \(V^{**}\) is an isometry of \(V\) onto the range of this map.

In general, this embedding is not surjective; if it is, then \((V, \| \cdot \|)\) is called reflexive

In finite dimensions, dimension arguments imply this map is surjective. Thus the dual of the dual norm is the original norm on \(V\).
Adjoint Transformations

The adjoint transformation of \( L \in \mathcal{L}(V, W) \), is

\[
(L^* g)(v) = g(Lv) .
\]

**Proposition.** Let \( V, W \) be normed linear spaces. If \( L \in \mathcal{B}(V, W) \), then

\[
L^*[W^*] \subset V^* .
\]

Moreover,

\[
L^* \in \mathcal{B}(W^*, V^*) \quad \text{and} \quad ||L^*|| = ||L|| .
\]

**Proof.** For \( g \in W^* \),

\[
|(L^* g)(v)| = |g(Lv)| \leq ||g|| \cdot ||L|| \cdot ||v|| ,
\]

so \( L^* g \in V^* \), and

\[
||L^* g|| \leq ||g|| \cdot ||L|| .
\]

Thus \( L^* \in \mathcal{B}(W^*, V^*) \) and \( ||L^*|| \leq ||L|| . \) Now given \( v \in V \), apply the Hahn-Banach theorem to \( Lv \) to conclude that

\[
\exists g \in W^* \text{ with } ||g|| = 1 \text{ and } (L^* g)(v) = g(Lv) = ||Lv|| .
\]

So

\[
||L^*|| = \sup_{||f|| \leq 1} ||L^* f||
\]

\[
= \sup_{||f|| \leq 1} \sup_{||v|| \leq 1} |(L^* f)(v)|
\]

\[
\geq \sup_{||v|| \leq 1} |(L^* g)(v)|
\]

\[
\geq \sup_{||v|| \leq 1} ||Lv|| = ||L|| .
\]

Hence \( ||L^*|| = ||L|| . \) 
\( \square \)
Completeness of $\mathcal{B}(V,W)$

**Proposition.** If $W$ is complete, then $\mathcal{B}(V,W)$ is complete.

In particular, $V^*$ is always complete (since $\mathbb{F}$ is), whether or not $V$ is.

**Proof.** If $\{L_n\}$ is Cauchy in $\mathcal{B}(V,W)$, then

$$(\forall v \in V) \quad \{L_n v\} \text{ is Cauchy in } W,$$

so the limit

$$\lim_{n \to \infty} L_n v = Lv$$

exists in $W$.

Clearly $L : V \to W$ is linear and $L \in \mathcal{B}(V,W)$.

To see that $\|L_n - L\| \to 0$, let $\epsilon > 0$ and choose $N$ such that $\|L_n - L_m\| \leq \epsilon$ $(\forall n, m \geq N)$. Then

$$\|L_n v - L_m v\| \leq \epsilon \quad (\forall v \in B, \ n, m \geq N).$$

Taking the limit in $m$ gives

$$\|L_n v - Lv\| \leq \epsilon \quad (\forall v \in B, \ n \geq N),$$

which shows

$$\|L_n - L\| \to 0 .$$
Analysis with Operators

$V$ a Banach space.

$V$ complete $\Rightarrow \mathcal{B}(V) = \mathcal{B}(V, V)$ complete (in the operator norm).

Let $U, V, W$ be normed linear spaces.

If $L \in \mathcal{B}(U, V)$ and $M \in \mathcal{B}(V, W)$, then for $u \in U$,

$$\|(M \circ L)(u)\|_W = \|M(Lu)\|_W \leq \|M\| \cdot \|Lu\|_V \leq \|M\| \cdot \|L\| \cdot \|u\|_U,$$

so $M \circ L \in \mathcal{B}(U, W)$ and

$$\|M \circ L\| \leq \|M\| \cdot \|L\|.$$

Therefore, Operator norms are always submultiplicative.

The special case $U = V = W$ shows that the operator norm on $\mathcal{B}(V)$ is submultiplicative so $\mathcal{B}(V)$ is an algebra:

$$L, M \in \mathcal{B}(V) \Rightarrow M \circ L \in \mathcal{B}(V).$$

We can define functions on $\mathcal{B}(V)$ using power series!

Given $L \in \mathcal{B}(V)$ and a polynomial $p(z) = a_0 + a_1z + \cdots + a_nz^n$

$$p(L) = a_0 I + a_1 L + \cdots + a_n L^n$$

is a polynomial mapping $\mathcal{B}(V)$ to itself, where

$L^k = L \circ \cdots \circ L$ and $L^0 \equiv I.$
Power Series on $\mathcal{B}(V)$

By taking limits of polynomials, we can form power series, and thus analytic functions on $\mathcal{B}(V)$.

For example, consider the series

$$e^L = \sum_{k=0}^{\infty} \frac{1}{k!} L^k = I + L + \frac{1}{2} L^2 + \cdots .$$

This series converges in the operator norm on $\mathcal{B}(V)$:

$$\|L^k\| \leq \|L\|^k,$$

so

$$\sum_{k=0}^{\infty} \frac{1}{k!} \|L^k\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|L\|^k$$

$$= e^{\|L\|} < \infty .$$

Since the series converges absolutely and $\mathcal{B}(V)$ is complete, it converges in the operator norm to an operator in $\mathcal{B}(V)$.

We call this operator $e^L$:

$$\|e^L\| \leq e^{\|L\|} .$$

In finite dimensions, this says that for $A \in \mathbb{F}^{n \times n}$, each component of the partial sum $\sum_{k=0}^{N} \frac{1}{k!} A^k$ converges as $N \to \infty$; the limiting matrix is $e^A$. 
Neumann Series

Consider the power series

\[ \frac{1}{1 - z} = \sum_{k=0}^{\infty} z^k \]

with radius of convergence 1 at \( z = 0 \).

**Lemma.** (C. Neumann)

If \( L \in \mathcal{B}(V) \) and \( \|L\| < 1 \), then \((I - L)^{-1}\) exists, and the Neumann series \( \sum_{k=0}^{\infty} L^k \) converges in the operator norm to \((I - L)^{-1}\).

An operator in \( \mathcal{B}(V) \) is said to be invertible if it is bijective \textit{and} its inverse is also in \( \mathcal{B}(V) \).

The Neumann Lemma is a very useful fact in the context of the perturbation theory for linear operators.
Proof. If \( \|L\| < 1 \), then
\[
\sum_{k=0}^{\infty} \|L^k\| \leq \sum_{k=0}^{\infty} \|L\|^k = \frac{1}{1 - \|L\|} < \infty,
\]
so the Neumann series \( \sum_{k=0}^{\infty} L^k \) converges to an operator in \( B(V) \).

Now if \( S_j, S, T \in B(V) \) and \( S_j \to S \) in \( B(V) \), then \( \|S_j - S\| \to 0 \), so
\[
\|S_jT - ST\| \leq \|S_j - S\| \cdot \|T\| \to 0
\]
and
\[
\|TS_j - TS\| \leq \|T\| \cdot \|S_j - S\| \to 0,
\]
and thus \( S_jT \to ST \) and \( TS_j \to TS \) in \( B(V) \).

Thus
\[
(I - L) \left( \sum_{k=0}^{\infty} L^k \right) = \lim_{N \to \infty} (I - L) \sum_{k=0}^{N} L^k
\]
\[
= \lim_{N \to \infty} (I - L^{N+1}) = I
\]
(as \( \|L^{N+1}\| \leq \|L\|^{N+1} \to 0 \)), and similarly
\[
\left( \sum_{k=0}^{\infty} L^k \right) (I - L) = I.
\]

So \( I - L \) is invertible and \( (I - L)^{-1} = \sum_{k=0}^{\infty} L^k \). \( \square \)
The power series arguments used above can be generalized.

Let \( f(z) \) be analytic on the disk \( \{ |z| < R \} \subset \mathbb{C} \), with power series

\[
f(z) = \sum_{k=0}^{\infty} a_k z^k
\]

and radius of convergence at least \( R \). If \( L \in \mathcal{B}(V) \) and \( ||L|| < R \), then the series

\[
\sum_{k=0}^{\infty} a_k L^k
\]

converges absolutely, and thus converges to an element of \( \mathcal{B}(V) \) which we call \( f(L) \).

It is easy to check that usual operational properties hold, for example

\[
(fg)(L) = f(L)g(L) = g(L)f(L).
\]

**Caution:**
Always remember that operators do not commute in general. For example,

\[
e^{L+M} \neq e^L e^M \quad \text{in general},
\]

although if \( L \) and \( M \) commute (i.e. \( LM = ML \)), then

\[
e^{L+M} = e^L e^M.
\]