Linear Analysis
Lecture 8

Adjoint Transformations

The adjoint transformation of $L \in \mathcal{L}(V, W)$, is

$$(L^* g)(v) = g(Lv) \ .$$

Proposition. Let V, W be normed linear spaces. If $L \in \mathcal{B}(V, W)$, then

$$L^*[W^*] \subset V^* \ .$$

Moreover, $L^* \in \mathcal{B}(W^*, V^*)$ and $\|L^*\| = \|L\|$.

Proof: For $g \in W^*$,

$$|(L^* g)(v)| = |g(Lv)| \leq \|g\| \cdot \|L\| \cdot \|v\|,$$

so $L^* g \in V^*$, and $\|L^* g\| \leq \|g\| \cdot \|L\|$.
Thus $L^* \in \mathcal{B}(W^*, V^*)$ and $\|L^*\| \leq \|L\|$. Now given $v \in V$, apply the Hahn-Banach theorem to Lv to conclude that

$$\exists\, g_v \in W^* \text{ with } \|g_v\| = 1 \text{ and } (L^* g_v)(v) = g_v(Lv) = \|Lv\|.$$

So

$$\begin{aligned} \|L^*\| = \sup_{\|f\| \leq 1} \|L^* f\| \quad &= \sup_{\|f\| \leq 1} \sup_{\|v\| \leq 1} |(L^* f)(v)| \\ \geq \sup_{\|v\| \leq 1} |(L^* g_v)(v)| \quad &= \sup_{\|v\| \leq 1} \|Lv\| = \|L\|. \end{aligned}$$

Hence $\|L^*\| = \|L\|$. $\square$

Completeness of $\mathcal{B}(V, W)$

Proposition. If W is complete, then $\mathcal{B}(V, W)$ is complete. In particular, V^* is always complete (since $\mathbb{F}$ is), whether or not V is complete.

Proof: If $\{L_n\}$ is Cauchy in $\mathcal{B}(V, W)$, then

$$(\forall\, v \in V) \quad \{L_n v\} \quad \text{is Cauchy in } W,$$

so the limit

$$\lim_{n\to\infty} L_n v \equiv Lv$$

exists in W.
Clearly $L : V \to W$ is linear and $L \in \mathcal{B}(V, W)$. To see that $\|L_n - L\| \to 0$, let $\epsilon > 0$ and choose N such that $\|L_n - L_m\| \le \epsilon \quad (\forall\, n, m \ge N)$. Then

$$\|L_n v - L_m v\| \le \epsilon \quad (\forall\, v \in B,\ n, m \ge N).$$

Taking the limit in m gives

$$\|L_n v - Lv\| \le \epsilon \quad (\forall\, v \in B,\ n \ge N),$$

which shows $\|L_n - L\| \to 0$. □

Let $(V, \|\cdot\|)$ be a Banach space. Since V complete, $\mathcal{B}(V) = \mathcal{B}(V, V)$ is complete in the operator norm.

Let U, V, W be normed linear spaces. If $L \in \mathcal{B}(U, V)$ and $M \in \mathcal{B}(V, W)$, then for $u \in U$,

$$\|(M \circ L)(u)\|_W = \|M(Lu)\|_W \leq \|M\| \cdot \|Lu\|_V \leq \|M\| \cdot \|L\| \cdot \|u\|_U,$$

so $\quad M \circ L \in \mathcal{B}(U, W) \quad$ and $\quad \|M \circ L\| \leq \|M\| \cdot \|L\|$.

Therefore, **Operator norms are always submultiplicative.**

The special case $U = V = W$ shows that the operator norm on $\mathcal{B}(V)$ is submultiplicative so $\mathcal{B}(V)$ is an algebra:

$$L, M \in \mathcal{B}(V) \Rightarrow M \circ L \in \mathcal{B}(V).$$

Polynomials on $\mathcal{B}(V)$.

Given $L \in \mathcal{B}(V)$ and a polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$,

$$p(L) \equiv a_0 I + a_1 L + \cdots + a_n L^n$$

is a polynomial mapping $\mathcal{B}(V)$ to itself, where

$$L^k = L \circ \cdots \circ L \quad \text{and} \quad L^0 \equiv I.$$

Power Series on $\mathcal{B}(V)$

By taking limits of polynomials, we can form power series, and thus analytic functions on $\mathcal{B}(V)$. For example, consider the series

$$e^L = \sum_{k=0}^{\infty} \frac{1}{k!} L^k = I + L + \frac{1}{2} L^2 + \cdots \ .$$

This series converges in the operator norm on $\mathcal{B}(V)$:

$$\|L^k\| \leq \|L\|^k,$$

so

$$\sum_{k=0}^{\infty} \frac{1}{k!} \|L^k\| \ \leq \ \sum_{k=0}^{\infty} \frac{1}{k!} \|L\|^k \ = \ e^{\|L\|} < \infty \ .$$

Since the series converges absolutely and $\mathcal{B}(V)$ is complete, it converges in the operator norm to an operator in $\mathcal{B}(V)$.

We call this operator e^L: $\|e^L\| \leq e^{\|L\|}$.

In finite dimensions, this says that for $A \in \mathbb{F}^{n \times n}$, each component of the partial sum $\sum_{k=0}^{N} \frac{1}{k!} A^k$ converges as $N \to \infty$; the limiting matrix is e^A.

Neumann Series

Consider the power series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

with radius of convergence 1 at $z = 0$.

Lemma.(C. Neumann)
If $L \in \mathcal{B}(V)$ and $\|L\| < 1$, then $(I - L)^{-1}$ exists, and the Neumann series $\sum_{k=0}^{\infty} L^k$ converges in the operator norm to $(I - L)^{-1}$.

An operator in $\mathcal{B}(V)$ is said to be invertible if it is bijective and its inverse is also in $\mathcal{B}(V)$.

The Neumann Lemma is an enormously useful fact in a number of contexts, e.g., the perturbation theory for linear operators.

Proof of Neumann's Lemma

If $\|L\| < 1$, then

$$\sum_{k=0}^{\infty} \|L^k\| \leq \sum_{k=0}^{\infty} \|L\|^k = \frac{1}{1 - \|L\|} < \infty,$$

so the Neumann series $\sum_{k=0}^{\infty} L^k$ converges to an operator in $\mathcal{B}(V)$.
Now if $S_j, S, T \in \mathcal{B}(V)$ and $S_j \to S$ in $\mathcal{B}(V)$, then $\|S_j - S\| \to 0$, so

$$\|S_j T - ST\| \leq \|S_j - S\| \cdot \|T\| \to 0$$

and

$$\|TS_j - TS\| \leq \|T\| \cdot \|S_j - S\| \to 0,$$

and thus $S_j T \to ST$ and $TS_j \to TS$ in $\mathcal{B}(V)$. Thus

$$(I - L)\left(\sum_{k=0}^{\infty} L^k\right) = \lim_{N\to\infty} (I - L) \sum_{k=0}^{N} L^k = \lim_{N\to\infty} (I - L^{N+1}) = I$$

(as $\|L^{N+1}\| \leq \|L\|^{N+1} \to 0$), and similarly

$$\left(\sum_{k=0}^{\infty} L^k\right)(I - L) = I.$$

So $(I - L)$ is invertible and $(I - L)^{-1} = \sum_{k=0}^{\infty} L^k$. $\square$

Power Series on $\mathcal{B}(V)$

Let $f(z)$ be analytic on the disk $\{|z| < R\} \subset \mathbb{C}$, with power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

and radius of convergence at least R. If $L \in \mathcal{B}(V)$ and $\|L\| < R$, then the series

$$\sum_{k=0}^{\infty} a_k L^k$$

converges absolutely, and thus converges to an element of $\mathcal{B}(V)$ which we call $f(L)$.

It is easy to check that usual operational properties hold, for example

$$(fg)(L) = f(L)g(L) = g(L)f(L) \ .$$

Caution: Always remember that operators do not commute in general. For example,

$$e^{L+M} \neq e^L e^M \qquad \text{in general,}$$

although if L and M commute (i.e. $LM = ML$), then

$$e^{L+M} = e^L e^M.$$

Calculus on $\mathcal{B}(V)$: Differentiation

Let $L(t)$ be a 1-parameter family of operators in $\mathcal{B}(V)$, $t \in (a, b)$. Since $\mathcal{B}(V)$ is a metric space, continuity of $L(t)$ in t is well defined.
We say that $L(t)$ is *differentiable* at $t = t_0 \in (a, b)$ if

$$L'(t_0) = \lim_{t \to t_0} \frac{L(t) - L(t_0)}{t - t_0}$$

exists in the norm on $\mathcal{B}(V)$.

For example, it is easily checked that for $L \in \mathcal{B}(V)$, e^{tL} is differentiable in t for all $t \in \mathbb{R}$, and

$$\frac{d}{dt} e^{tL} = L e^{tL} = e^{tL} L \ .$$

Calculus on $\mathcal{B}(V)$: Differentiation

Similarly consider families of operators in $\mathcal{B}(V)$ depending on several real or complex parameters. A family $L(z)$ where

$$z = x + iy \in \Omega^{\text{open}} \subset \mathbb{C} \qquad (x, y \in \mathbb{R})$$

is said to be holomorphic in Ω if the partial derivatives

$$\frac{\partial}{\partial x} L(z), \qquad \frac{\partial}{\partial y} L(z)$$

exist, are continuous in Ω, and satisfy the Cauchy-Riemann equations

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) L(z) = 0 \quad \text{on } \Omega.$$

This is equivalent to the assumption that in a neighborhood of each point $z_0 \in \Omega$, $L(z)$ is given by the $\mathcal{B}(V)$-norm convergent power series

$$L(z) = \sum_{k=0}^{\infty} \frac{1}{k!} (z - z_0)^k \left(\frac{d}{dz} \right)^k L(z_0).$$

Calculus on $\mathcal{B}(V)$: Integration

If $L(t)$ depends continuously on $t \in [a, b]$, then it can be shown using the same estimates as for $\mathbb{F}$-valued functions (and the uniform continuity of $L(t)$ since $[a, b]$ is compact) that the Riemann sums

$$\frac{b-a}{N} \sum_{k=0}^{n-1} L\left(a + \frac{k}{N}(b-a)\right)$$

converge in $\mathcal{B}(V)$-norm (recall V is a Banach space) as $n \to \infty$ to an operator in $\mathcal{B}(V)$, denoted

$$\int_a^b L(t)\, dt \; .$$

More general Riemann sums than just the left-hand "rectangular rule" with equally spaced points can be used.

Many results from standard calculus carry over, including

$$\left\| \int_a^b L(t)\, dt \right\| \leq \int_a^b \|L(t)\|\, dt$$

which follows from

$$\left\| \frac{b-a}{N} \sum_{k=0}^{N-1} L\left(a + \frac{k}{N}(b-a)\right) \right\| \leq \frac{b-a}{N} \sum_{k=0}^{N-1} \left\| L\left(a + \frac{k}{N}(b-a)\right) \right\| .$$

By parameterizing paths in $\mathbb{C}$, one can define line integrals of holomorphic families of operators. We will discuss such constructions further as we need them.

Transposes and Adjoints

$$\begin{array}{rcll} A & \in & \mathbb{C}^{m\times n} & \\ A^T & \in & \mathbb{C}^{n\times m} & \text{the transpose of } A\text{: } (A_{ij} = (A^T)_{ji} \\ A^H & = & \bar{A}^T & \text{the conjugate-transpose (or Hermitian transpose) of } A \end{array}$$

One often writes $A^* = A^H$. But there is a subtlety here, which is related to the identification of a linear operator with its representation as a matrix, that we must be careful about.

Recall that an inner product on $\mathbb{C}^n$ can be represented as a matrix multiply, e.g. for the Euclidean inner product

$$\langle x, y\rangle = y^H x$$

For $A \in \mathbb{C}^{n\times n}$,

$$\langle Ax, y\rangle = \langle x, A^H y\rangle$$

since

$$y^H Ax = (A^H y)^H x$$

Caution About A^H and A^*

The notation A^* is used with two different meanings (particularly when $\mathbb{F} = \mathbb{C}$).
$$L \in \mathcal{B}(V, W) \implies L^* \in \mathcal{B}(W^*, V^*)$$
In finite dim., one can choose bases of V and W to encode the action of L as left matrix multiplication on column vectors associated with the bases components. Denote such a matrix as T.
Then the action of L^* on W^* can be encoded using the corresponding dual bases *using the same matrix* T. But now the action is represented as *right* multiplication by T on row vectors of components in the dual basis (a=bT).
On the other hand, in the presence of an inner product, the definition
$$\langle Lv, w \rangle = \langle v, L^* w \rangle$$
identifies L^* with left-multiplication by the conjugate-transpose matrix.
These two definitions are related by the identification
$$V \cong V^*$$
induced by the inner product:
$$w \in V \iff \langle \cdot, w \rangle = w^* \in V^* \ .$$
But the conjugation in this identification gives rise to a different representation of L^*. The first is a natural representation obtained through the composition of function, and the second is through the inner product.

Norms on Matrices

Commonly used norms on $\mathbb{C}^{m\times n}$.

$$\|A\|_1 = \sum_{ij} |a_{ij}| \qquad \text{(the } \ell^1\text{-norm on } A \text{ as if it were in } \mathbb{C}^{n^2}\text{)}$$

$$\|A\|_\infty = \max_{i,j} |a_{ij}| \qquad \text{(the } \ell^\infty\text{-norm on } A \text{ as if it were in } \mathbb{C}^{n^2}\text{)}$$

$$\|A\|_2 = \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}} \qquad \text{(the } \ell^2\text{-norm on } A \text{ as if it were in } \mathbb{C}^{n^2}\text{)}$$

$\|A\|_2$ is called the Hilbert-Schmidt norm of A, or the Frobenius norm of A, and is often denoted $\|A\|_F$. It is also called the Euclidean norm on $\mathbb{C}^{m\times n}$. The associated inner product is $\langle A, B\rangle = \operatorname{tr} B^H A$.

We also have p-norms for matrices: let $1 \le p \le \infty$,

$$|||A|||_p = \max_{\|x\|_p=1} \|Ax\|_p \quad \left(= \max_{\|x\|_p\le 1} \|Ax\|_p = \max_{x\neq 0}(\|Ax\|_p/\|x\|_p)\right) .$$

In particular,

$$|||A|||_1 = \max_{1\le j\le n} \sum_{i=1}^m |a_{ij}| \qquad \text{(max column sum)}$$

$$|||A|||_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}| \qquad \text{(max row sum)}$$

$|||A|||_2$ is the spectral norm. We study it in detail later.

Consistent Matrix Norms

Definitions

• Let $\mu : \mathbb{C}^{m\times n} \to \mathbb{R}$, $\nu : \mathbb{C}^{n\times k} \to \mathbb{R}$, $\rho : \mathbb{C}^{m\times k} \to \mathbb{R}$ be norms. We say that μ, ν, ρ are **consistent** if $\forall\, A \in \mathbb{C}^{m\times n}$ and $\forall\, B \in \mathbb{C}^{n\times k}$,

$$\rho(AB) \leq \mu(A)\nu(B)$$

• A norm on $\mathbb{F}^{n\times n}$ is called consistent if it is consistent with itself, i.e., the definition above with $m = n = k$ an $\rho = \mu = \nu$. So by definition a norm on $\mathbb{F}^{n\times n}$ is consistent iff it is submultiplicative.

• A collection $\{\nu_{m,n} : m \geq 1, n \geq 1\}$, where $\nu_{m,n} = \mathbb{F}^{m\times n} \to \mathbb{R}$ is a norm on $\mathbb{F}^{m\times n}$, is called a **family of matrix norms**.

• A family $\{\nu_{m,n} : m \geq 1, n \geq 1\}$ of matrix norms is called **consistent** if

$$(\forall\, m, n, k \geq 1)(\forall\, A \in \mathbb{F}^{m\times n})(\forall\, B \in \mathbb{F}^{n\times k})$$
$$\nu_{m,k}(AB) \leq \nu_{m,n}(A)\nu_{n,k}(B).$$

Facts Let $\{\nu_{m,n}\}$ be a consistent family of matrix norms. Then

(1) $(\forall\, n \geq 1) \quad \nu_{n,n}$ is submultiplicative.

(2) $(\forall\, m, n \geq 1)\ (\forall\, A \in \mathbb{F}^{m\times n}) \quad \nu_{m,n}(A) \geq \mu_{m,n}(A)$, where $\mu_{m,n}$ is the operator norm on $\mathbb{F}^{m\times n}$ induced by $\nu_{n,1}$ and $\nu_{m,1}$.

Examples of Consistent Matrix Norms

(1) For $m > 1$, let $\nu_{m,1}$ be any norm on $\mathbb{F}^m$, and $\nu_{1,1}(x) = |x|$. For $m, n \geq 1$, let $\nu_{m,n}$ be the operator norm on $\mathbb{F}^{m\times n}$ induced by $\nu_{n,1}$ and $\nu_{m,1}$ Then $\{\nu_{m,n}\}$ is a consistent family of matrix norms.

(2) (maximum row sum norm) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let

$$\nu_{m,n}(A) = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|.$$

Then $\nu_{n,1}$ is the ℓ^∞-norm on $\mathbb{F}^n$, and $\nu_{m,n}(A)$ is the operator norm induced by the ℓ^∞-norms on $\mathbb{F}^n$ and $\mathbb{F}^m$, which we denoted by $|||A|||_\infty$.

(3) (maximum column sum norm) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let

$$\nu_{m,n}(A) = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|.$$

Then $\nu_{n,1}$ is the ℓ^1-norm on $\mathbb{F}^n$, and $\nu_{m,n}(\cdot)$ is the operator norm induced by the ℓ'-norms on $\mathbb{F}^n$ and $\mathbb{F}^m$, which we denoted by $|||A|||_1$.

(4) (ℓ^1-norm on $\mathbb{F}^{m\times n}$ as if it were $\mathbb{F}^{mn}$) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let

$$\nu_{m,n}(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|.$$

Then $\{\nu_{m,n}\}$ is a consistent family of matrix norms. We denoted $\nu_{m,n}(A)$ by $\|A\|_1$. Also note that $\|A\|_1 \geq |||A|||_1$ agrees with Fact (2) on the previous slide.

The Frobenius or Hilbert-Schmidt Norm

(ℓ^2-norm on $\mathbb{F}^{m\times n}$ as if it were $\mathbb{F}^{mn}$) For $m, n \geq 1$ and $A \in \mathbb{F}^{m\times n}$, let

$$\nu_{m,n}(A) = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}.$$

Then $\nu_{n,1}$ is the ℓ^2-norm on $\mathbb{F}^n$. If $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times k}$, then by the Schwarz inequality,

$$\begin{aligned}(\nu_{m,k}(AB))^2 &= \sum_{i=1}^{m}\sum_{j=1}^{k}\left|\sum_{\ell=1}^{n} a_{i\ell} b_{\ell_j}\right|^2 \\ &\leq \sum_{i=1}^{m}\sum_{j=1}^{k}\left(\sum_{\ell=1}^{n}|a_{i\ell}|^2\right)\left(\sum_{r=1}^{n}|b_{rj}|^2\right) \\ &= (\nu_{m,n}(A)\nu_{n,k}(B))^2,\end{aligned}$$

so $\{v_{m,n}\}$ is a consistent family of matrix norms.
This is **not** an operator norm: for $n > 1$, $\nu_{n,n}(I) = \sqrt{n}$ but the operator norm of I is always 1.

Denote $\nu_{m,n}(A)$ by $\|A\|_2$ (or, sometimes $\|A\|_F$).

The Frobenius or Hilbert-Schmidt Norm

For $A \in \mathbb{F}^{m \times n}$ and $x \in \mathbb{F}^n$, we have

$$\|Ax\|_2 \leq \|A\|_2 \cdot \|x\|_2,$$

so for $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times k}$,

$$\|AB\|_2 \leq \|A\|_2 \cdot \|B\|_2.$$

Fact (2) above gives the important inequality

$$\|A\|_F \geq \|\|A\|\|_2 \qquad \forall\ A \in \mathbb{F}^{m \times n}.$$

Thus the operator norm induced by the ℓ^2-norms on $\mathbb{F}^m$ and $\mathbb{F}^n$ is dominated by the Frobenius norm.

Condition Number and Error Sensitivity

Let $A \in \mathbb{C}^{n\times n}$ be invertible. How sensitive is the solution to $Ax = b$ to changes in b and A? Let $\|\cdot\|$ be a consistent martrix norm on $\mathbb{C}^{n\times n}$. Suppose $Ax = b$ and $A\hat{x} = \hat{b}$. How far is $\hat{x}$ from x?

$$\begin{aligned} e &:= x - \hat{x} = \text{ the error vector} \\ \|e\| &:= \text{the error} \\ r &:= b - \hat{b} = \text{ the residual vector} \\ \|r\| &:= \text{the residual} \\ \|e\|/\|x\| &:= \text{the relative error} \\ \|r\|/\|b\| &:= \text{the relative residual} \end{aligned}$$

Then $Ae = A(x - \hat{x}) = b - \hat{b} = r$, so

$$\|e\| = \|A^{-1}r\| \leq \|A^{-1}\|\,\|r\| \quad \text{and} \quad \|b\| \leq \|A\|\,\|x\|$$

so

$$\|e\| \cdot \|b\| \leq \|A\| \cdot \|A^{-1}\| \cdot \|x\| \cdot \|r\|$$

which implies

$$\frac{\|e\|}{\|x\|} \leq \kappa(A)\frac{\|r\|}{\|b\|},$$

where $\kappa(A) := \|A\| \cdot \|A^{-1}\|$ is the condition number of A.

Condition Number and Error Sensitivity

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = \text{condition number of } A.$$

Note that for any operator norm

$$1 = \|I\| = \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\| = \kappa(A).$$

A matrix is said to be perfectly conditioned if $\kappa(A) = 1$, and is said to be ill-conditioned if $\kappa(A)$ is large.

If $\widehat{x}$ is the result of a numerical algorithm for solving $Ax = b$ (with round-off error), then the error $e = x - \widehat{x}$ is not computable. However, the residual $r = b - A\widehat{x}$ is computable, so we obtain an upper bound on the relative error

$$\frac{\|e\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|} \ .$$

In practice, we don't know $\kappa(A)$ (although we may be able to estimate it), and this upper bound may be much larger than the actual relative error.

Condition Number and Error Sensitivity

Now assume that both A and b are perturbed:

$$\hat{A} = A + E \quad \text{and} \quad \hat{b} = b - r$$

Obtain a bound on the relative error $\|e\|/\|x\|$ where

$$e = x - \hat{x} \quad \text{with } \hat{x} \text{ solving } \hat{A}\hat{x} = \hat{b} \ .$$

First use the Neumann Lemma to note that if

$$\|A^{-1}E\| \le \|A^{-1}\| \, \|E\| < 1,$$

then

$$\begin{aligned} \|(A+E)^{-1}\| &= \|(I + A^{-1}E)^{-1}A^{-1}\| \\ &\le \|(I + A^{-1}E)^{-1}\| \, \|A^{-1}\| \quad \le \frac{1}{1 - \|A^{-1}\| \, \|E\|} \|A^{-1}\| \ . \end{aligned}$$

Also, $(A+E)x = b + Ex$ and $(A+E)\widehat{x} = \widehat{b}$ so

$$e = x - \hat{x} = (A+E)^{-1}(Ex + r)$$

and so

$$\begin{aligned} \|e\| &\le \|(A+E)^{-1}\| \left[\|E\| \, \|x\| + \|r\|\right] \\ &\le \frac{1}{1 - \|A^{-1}\| \ \|E\|} \|A^{-1}\| \ \left[\frac{\|E\|}{\|A\|}\|A\| \, \|x\| + \frac{\|r\|}{\|b\|}\|A\| \, \|x\|\right] \ . \end{aligned}$$

Therefore,

$$\frac{\|e\|}{\|x\|} \le \frac{\kappa(A)}{1 - \kappa(A)\frac{\|E\|}{\|A\|}} \left[\frac{\|E\|}{\|A\|} + \frac{\|r\|}{\|b\|}\right] \ .$$