Linear Analysis
Lecture 7

Norms on Operators

If V, W are vector spaces, then so is the space of linear transformations from V to W denoted $\mathcal{L}(V, W)$. When $V = W$, $\mathcal{L}(V, V) = \mathcal{L}(V)$ is an algebra with composition as multiplication.

Norms on $\mathcal{L}(V)$ compatible with composition are particularly useful. A norm on $\mathcal{L}(V)$ is said to be it submultiplicative if

$$\|A \circ B\| \leq \|A\| \cdot \|B\| \ .$$

Not all matrix norms are submultiplicative.

For $A \in \mathbb{C}^{n \times n}$, define

$$\|A\| = \sup_{1 \leq i,j \leq n} |a_{ij}| \ .$$

Then, if

$$A = B = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

then $\|A\| = \|B\| = 1$, but $AB = A^2 = nA$ so $\|AB\| = n$.

But it can be shown that the norm

$$A \mapsto n \sup_{1 \leq i,j \leq n} |a_{ij}|$$

is submultiplicative.

Bounded Linear Operators

Let $(V, \|\cdot\|_v)$ and $(W, \|\cdot\|_w)$ be normed linear spaces.
$L \in \mathcal{L}(V, W)$ is called a **bounded linear operator** if

$$\sup_{\|v\|_v=1} \|Lv\|_w < \infty \ .$$

$\mathcal{B}(V, W)$ denotes the set of all bounded linear operators from V to W.

If $W = \mathbb{F}$, this gives the set of **bounded linear functionals**, and we set

$$V^* = \mathcal{B}(V, \mathbb{F}) \ .$$

If $\dim V < \infty$, then $\mathcal{L}(V, W) = \mathcal{B}(V, W)$, so also $V^* = V'$.

Not all linear operators are bounded.

Let $V = \mathcal{P}$ be the space of polynomials with norm

$$\|p\| = \sup_{0 \le x \le 1} |p(x)|.$$

Then $\frac{d}{dx} : \mathcal{P} \to \mathcal{P}$ is not a bounded linear operator:

$$\|x^n\| = 1 \quad \text{for all} \quad n \ge 1 \quad \text{but} \quad \left\| \frac{d}{dx} x^n \right\| = \|n x^{n-1}\| = n$$

Operator Norms

Definition. Let $L \in \mathcal{B}(V, W)$. The operator norm of L is

$$\|L\| = \sup_{\|v\|_v \leq 1} \|Lv\|_w.$$

This makes $\mathcal{B}(V, W)$ a normed linear space.
In the special case $W = \mathbb{F}$, the norm

$$\|f\|_* = \sup_{\|v\| \leq 1} |f(v)|$$

on V^* is called the **dual norm**.
Therefore,

$$|f(v)| \leq \|f\|_* \, \|v\|, \quad \forall \; v \in V, \; f \in V^*.$$

If $\dim V < \infty$, choose bases to identify V and V^* with $\mathbb{F}^n$. Thus, every norm $\|\cdot\|$ on $\mathbb{F}^n$ has a dual norm $\|\cdot\|_*$ on $\mathbb{F}^n$ satisfying

$$|\langle v, w \rangle| \leq \|v\| \, \|w\|_* \, .$$

We sometimes write $\mathbb{F}^{n^*}$ for $\mathbb{F}^n$ when it is being identified with V^*.

Duals of ℓ^p Norms

V a finite dimensional vector space with basis $\{v_1, \ldots, v_n\}$ and dual basis $\{f_1, \ldots, f_n\}$.
Let $v \in V$ and $f \in V^*$ have coordinates

$$v \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad f \mapsto (f_1, \ldots, f_n)\ , \quad \text{respectively.}$$

Given $1 \leq p \leq \infty$, the mapping

$$\|v\| = \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_p$$

defines a norm on V.
The norm dual to this norm is

$$\|f\|_* = \|(f_1, \ldots, f_n)^T\|_q\ ,$$

where

$$\frac{1}{p} + \frac{1}{q} = 1\ .$$

This is a consequence of Hölder's inequality.

Hyperplane Geometry and Duality

Let V be a vector space over $\mathbb{F}$. A **hyperplane** in V is a set of the form

$$H(f,c) \;=\; \{v \in V \,:\, f(v) = c\}\;, \quad \text{where } f \in V^* \text{ and } c \in \mathbb{F}.$$

If $c = 0$, then $H(f,0)$ is a subspace of codimension 1.
If $\mathbb{F} = \mathbb{C}$, it is often more desirable to use the real hyperplanes:

$$H_r(f,c) \;=\; \{v \in V \,:\, \mathcal{R}e(f(v)) = c\}\;,$$

Proposition. If $(V, \|\cdot\|)$ is a normed linear space and $f \in V^*$, then the dual norm of f satisfies

$$\|f\|_* = \sup_{\|v\| \le 1} \mathcal{R}e(f(v))\;.$$

Proof Since $\mathcal{R}e(f(v)) \le |f(v)|$,

$$\sup_{\|v\| \le 1} \mathcal{R}e(f(v)) \le \sup_{\|v\| \le 1} |f(v)| = \|f\|_*.$$

For the other direction, choose a sequence $\{v_j\}$ from V with $\|v_j\| = 1$ and $|f(v_j)| \to \|f\|_*$. Taking $\theta_j = -\arg f(v_j)$ and setting $w_j = e^{i\theta_j} v_j$, we have $\|w_j\| = 1$ and $f(w_j) = |f(v_j)| \to \|f\|_*$, so

$$\sup_{\|v\| \le 1} \mathcal{R}e(f(v)) \ge f(w_j) = |f(v_j)| \to \|f\|_*.$$

Hyperplane Geometry and Duality

The previous Proposition provides a dual description of the closed unit ball $\mathcal{B} \subset V^*$ as an intersection of **closed half-spaces**.
To see this, recall that every hyperplane defines two closed halfspaces

$$H_r^-(f,c) = \{v \in V : \mathcal{R}e(f(v)) \leq c\} \text{ and } H_r^+(f,c) = \{v \in V : \mathcal{R}e(f(v)) \geq c\}.$$

Let $\mathbb{B}^\circ$ denote the unit ball of the dual norm on V^*:

$$\mathbb{B}^\circ = \{f \in V^* : \|f\|_* \leq 1\} = \{f \in V^* : \mathcal{R}e(f(v)) \leq 1 \; \forall \, v \in \mathbb{B}\} \; .$$

The Proposition implies that given $v \in \mathbb{B}$,

$$\mathcal{R}e(f(v)) \leq \|f\| \leq 1 \quad \forall \, f \in \mathbb{B}^\circ.$$

or equivalently,

$$\mathbb{B} \subset \bigcap_{f \in \mathbb{B}^\circ} H_r^-(f,1) = \{v \in V : \mathcal{R}e(f(v)) \leq 1 \; \forall \, f \in \mathbb{B}^\circ\}.$$

Alternatively,

$$v \notin \mathbb{B} \quad \Leftrightarrow \quad \exists \, f \in \mathbb{B}^\circ \text{ such that } f(v) > 1.$$

Proposition.(Geometric Form of the Hahn-Banach Theorem)
Let C be a convex subset of the normed linear space $(V, \|\cdot\|)$ with non-empty interior. If $v \notin \text{int}\,(V)$, then there exists $f \in V^*$ such that

$$\mathcal{R}e(f(v)) \; > \; \sup_{u \in C} \mathcal{R}e(f(u)) \; .$$

Corollary. If $(V, \|\cdot\|)$ is a normed linear space and $v \in V$, then there exists $f \in V^*$ such that

$$f(v) = \|v\| \, \|f\| \; .$$

The Geometric Form of the Hahn-Banach Theorem is also known as the **fundamental separation theorem for convex sets** on a normed linear space. The linear functional f in the Proposition is said to **properly separate** v from the set C. Equivalently, f determines a a hyperplane such that v and $\text{int}\,(C)$ lie in opposing open half spaces.

Consequences of the Hahn-Banach Theorem for the 2nd Dual

Consider the normed linear space $(V, \|\cdot\|)$ and its dual $(V^*, \|\cdot\|)$.
Given $v \in V$, define $v^{**} \in V^{**}$ by

$$v^{**}(f) = f(v) \ .$$

Then

$$|v^{**}(f)| = |f(v)| \leq \|f\| \cdot \|v\| \quad \Longrightarrow \quad \|v^{**}\| \leq \|v\|$$

The Hahn-Banach theorem $\Rightarrow$

$$\exists\, f \in V^* \ \ni \ \|f\| = 1 \text{ and } v^{**}(f) = f(v) = \|v\| \ ,$$

so

$$\|v^{**}\| = \sup_{\|f\|=1} |v^{**}(f)| \geq \|v\| \ .$$

Hence $\|v^{**}\| = \|v\|$. That is, the mapping $v \mapsto v^{**}$ from V into V^{**} is an isometry of V onto the range of this map.
This embedding is not, in general, surjective; however, when it is, $(V, \|\cdot\|)$ is called a **reflexive Banach space**.
In finite dimensions, dimension arguments imply this map is surjective.
Thus the dual of the dual norm is the original norm on V.