Linear Analysis
Lecture 6

Examples

(1) The Euclidean norm [i.e. ℓ^2 norm] on $\mathbb{F}^n$ is induced by the standard inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i} \quad : \quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i \overline{x_i}} = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

(2) Let $A \in \mathbb{F}^{n\times n}$ be Hermitian-symmetric and positive definite, and let

$$\langle x, y\rangle_A = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} \overline{y_j} \quad \text{for } x, y \in \mathbb{F}^n.$$

Then $\langle \cdot, \cdot\rangle_A$ is an inner product on $\mathbb{F}^n$, which induces the norm

$$\|x\|_A = \sqrt{\langle x, x\rangle_A} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} \overline{x_j}} = \sqrt{x^T A \overline{x}} = \sqrt{x^H \overline{A} x}.$$

Examples

(3) The ℓ^2-norm on ℓ^2 (subspace of $\mathbb{F}^\infty$) is induced by the inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} \quad : \quad \|x\|_2 = \sqrt{\sum_{i=1}^{\infty} x_i \overline{x_i}} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}.$$

(4) The L^2 norm

$$\|u\|_2 = \left(\int_a^b |u(x)|^2\,dx\right)^{\frac{1}{2}}$$

on $C([a,b])$ is induced by the inner product

$$\langle u, v\rangle = \int_a^b u(x)\overline{v(x)}\,dx\ .$$

Convexity

A subset C of a vector space V is called *convex* if

$$(\forall\, v, w \in C)(\forall\, t \in [0,1]) \qquad tv + (1-t)u \in C.$$

Let $B = \{v \in V : \|v\| \leq 1\}$ denote the closed unit ball in a *finite dimensional* normed linear space.

Facts.

(1) B is convex.

(2) B is compact.

(3) B is symmetric (if $v \in B$ and $\alpha \in \mathbb{F}$ with $|\alpha| = 1$, then $\alpha v \in B$).

(4) The origin is in the interior of B.

Lemma. If $\dim V < \infty$ and $B \subset V$ satisfies the four conditions above, then there is a unique norm on V for which B is the closed unit ball:

$$\|v\| = \inf\{c > 0 : \frac{v}{c} \in B\}.$$

Completeness

$(V, \|\cdot\|)$ a normed linear space. We say $(V, \|\cdot\|)$ is *complete* if every Cauchy sequence in V has a limit in V.

- $\{v_n\} \subset V$ is Cauchy if $(\forall\, \epsilon > 0)(\exists\, N)(\forall\, n, m \geq N)\ \|v_n - v_m\| < \epsilon$.
- $\{v_n\} \subset V$ has limit v if $\|v - v_n\| \to 0$.

For example, $(\mathbb{F}^n,\ \|\cdot\|_2)$ is complete.

Topological properties are those that depend only on the collection of open sets (e.g., open, closed, compact, whether a sequence converges, etc.). Completeness is *not* a topological property.

Example: Let $f : [1, \infty) \to (0, 1]$ be given by $f(x) = \frac{1}{x}$ (with the usual metric on $\mathbb{R}$). Then f is a homeomorphism (bijective, bicontinuous), but $[1, \infty)$ *is* complete while $(0, 1]$ is *not* complete.

Completeness is a *uniform property*.

Theorem:*If* (X, ρ) *and* (Y, σ) *are metric spaces, and* $\varphi : (X, \rho) \to (Y, \sigma)$ *is a uniform homeomorphism (i.e., bijective, bicontinuous and* φ *and* φ^{-1} *are both uniformly continuous), then* (X, ρ) *is complete iff* (Y, σ) *is complete.*

Completeness

Since bounded linear operators between normed linear spaces are automatically uniformly continuous, several facts follow immediately.

Corollary. If two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are equivalent, then $(V, \|\cdot\|_1)$ is complete iff $(V, \|\cdot\|_2)$ is complete.

Corollary. Every finite-dim normed linear space is complete.

But not every infinite-dim normed linear space is complete.

Definition. A complete normed linear space is called a *Banach space*. An inner product space for which the induced norm is complete is called a *Hilbert space*.

To show that a normed linear space is complete, we must show that every Cauchy sequence converges in that space. The basic strategy for showing completeness is a three step process that can be described as follows: Given a Cauchy sequence,

(i) construct what you think is its limit;

(ii) show the limit is in the space V;

(iii) show the sequence converges to the limit in V.

Example: $(C_b(M), \|\cdot\|)$, M a metric space

$C(M)$ the vector space of continuous functions $u : M \to \mathbb{F}$.
$C_b(M)$ the subspace of $C(M)$ of all bounded continuous functions.
On $C_b(M)$, define the sup-norm $\|u\| = \sup_{x\in M} |u(x)|$.

Fact: $(C_b(M), \|\cdot\|)$ is complete.

Proof: Let $\{u_n\} \subset C_b(M)$ be Cauchy in $\|\cdot\|$. Given $\epsilon > 0$, $\exists\, N$ so that $(\forall\, n, m \geq N) \qquad \|u_n - u_m\| < \epsilon$. For each $x \in M$, $|u_n(x) - u_m(x)| \leq \|u_n - u_m\|$, so for each $x \in M$, $\{u_n(x)\}$ is a Cauchy sequence in $\mathbb{F}$, which has a limit in $\mathbb{F}$ (which we will call $u(x)$) since $\mathbb{F}$ is complete: $u(x) = \lim_{n\to\infty} u_n(x)$. Let $\epsilon > 0$, then

$$(\exists\, N)(\forall\, n, m \geq N)(\forall\, x \in M) \qquad |u_n(x) - u_m(x)| < \epsilon.$$

Take the limit (for each fixed x) to get

$$(\forall\, n \geq N)(\forall\, x \in M) \qquad |u_n(x) - u(x)| \leq \epsilon.$$

Thus $u_n \to u$ uniformly, so u is continuous (since the uniform limit of continuous functions is continuous). Clearly u is bounded, so $u \in C_b(M)$. We have $\|u_n - u\| \to 0$ as $n \to \infty$, i.e., $u_n \to u$ in $(C_b(M), \|\cdot\|)$. $\square$

ℓ^p is complete for $1 \le p \le \infty$

$\underline{p = \infty}$. This is a special case of $C_b(M)$ where $M = \mathbb{N} = \{1, 2, 3, \dots\}$.
$\underline{1 \le p < \infty}$. Let $\{x_k\} \subset \ell^p$ be Cauchy. Write $x_k = (x_{k1}, x_{k2}, \dots)$. Given $\epsilon > 0$, $\quad (\exists\, K)(\forall\, k, \ell \ge K) \quad \|x_k - x_\ell\|_p < \epsilon$. For each $m \in \mathbb{N}$,

$$|x_{km} - x_{\ell m}| \le \left(\sum_{i=1}^{\infty} |x_{ki} - x_{\ell i}|^p \right)^{\frac{1}{p}} = \|x_k - x_\ell\|,$$

so, for each $m \in \mathbb{N}$, $\{x_{km}\}_{k=1}^{\infty}$ is Cauchy with limit $x_m = \lim_{k\to\infty} x_{km}$.
Let x be the sequence $x = (x_1, x_2, x_3, \dots)$; so far, we know that $x \in \mathbb{F}^\infty$.
Given $\epsilon > 0$, $(\exists\, K)(\forall\, k, \ell \ge K)$ $\|x_k - x_\ell\| < \epsilon$. Then

$$\text{for any } N \text{ and for } k, \ell \ge K, \quad \left(\sum_{i=1}^{N} |x_{ki} - x_{\ell i}|^p \right)^{\frac{1}{p}} < \epsilon.$$

$$\text{Taking the limit in } \ell, \quad \left(\sum_{i=1}^{N} |x_{ki} - x_i|^p \right)^{\frac{1}{p}} \le \epsilon.$$

$$\text{Taking the limit in } N, \quad \left(\sum_{i=1}^{\infty} |x_{ki} - x_i|^p \right)^{\frac{1}{p}} \le \epsilon.$$

Thus $x_k - x \in \ell^p$, so also $x = x_k - (x_k - x) \in \ell^p$, and we have
$(\forall\, k \ge K)$ $\|x_k - x\|_p \le \epsilon$. Thus $\|x_k - x\|_p \overset{k\to\infty}{\to} 0$, i.e., $x_k \to x$ in ℓ^p. $\square$

$\mathbb{F}_0^\infty = \{x \in \mathbb{F}^\infty : (\exists\, N)(\forall\, n \geq N) \quad x_n = 0\}$

$\mathbb{F}_0^\infty$ is *not* complete in any ℓ^p norm $(1 \leq p \leq \infty)$.

<u>$1 \leq p < \infty$</u>.
Choose any $x \in \ell^p \backslash \mathbb{F}_0^\infty$, and consider the truncated sequences

$$y_1 = (x_1, 0, \ldots),\ \ y_2 = (x_1, x_2, 0, \ldots),\ \ y_3 = (x_1, x_2, x_3, 0, \ldots),\ \ \ldots$$

$\{y_n\}$ is Cauchy in $(\mathbb{F}_0^\infty, \|\cdot\|_p)$, but there is no $y \in \mathbb{F}_0^\infty$ for which $\|y_n - y\|_p \to 0$.

<u>$p = \infty$</u>.
Same idea: choose any

$$x \in \ell^\infty \backslash \mathbb{F}_0^\infty$$

for which

$$\lim_{i \to \infty} x_i = 0\ ,$$

and consider the sequence of truncated sequences.

Every Metric Space can be Completed

Fact

Let (X, ρ) be a metric space. Then there exists a complete metric space $(\bar{X}, \bar{\rho})$ and an "inclusion map" $i : X \to \bar{X}$ for which

- *i is injective,*
- *i is an isometry from X to $i[X]$, i.e.*

$$(\forall\, x, y \in X)\ \rho(x, y) = \bar{\rho}(i(x), i(y))),$$

  and
- *$i[X]$ is dense in $\bar{X}$.*

Moreover, all such $(\bar{X}, \bar{\rho})$ are isometrically isomorphic. The metric space $(\bar{X}, \bar{\rho})$ is called the completion of (X, ρ).

One way to construct such an $\bar{X}$ is to take equivalence classes of Cauchy sequences in X to be elements of $\bar{X}$.

Representations of Completions

Sometimes the completion of a metric space can be identified with a larger vector space which actually includes X, and whose elements are objects of a similar nature to the elements of X. One example is $\mathbb{R} =$ completion of the rationals $\mathbb{Q}$.

The completion of $C([a, b])$ in the L^p norm (for $1 \leq p < \infty$) is denoted by $L^p([a, b])$.

$L^p([a, b])$ is the vector space of equivalence classes of Lebesgue measurable functions $u : [a, b] \to \mathbb{F}$ for which $\int_a^b |u(x)|^p \, dx < \infty$, with norm

$$\|u\|_p = \left(\int_a^b |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Fact

A subset of a complete metric space is complete iff it is closed.

Proposition. Let V be a Banach space, and $W \subset V$ be a subspace. The norm on V restricts to a norm on W. We have:

W is complete iff W is closed.

Examples of Complete Spaces as Closed Subspaces

Consider the spaces $C_0(\mathbb{R}^n)$ and $C_c(\mathbb{R}^n)$.

$$\begin{aligned} C_0(\mathbb{R}^n) &= \{u \in C_b(\mathbb{R}^n) : \lim_{|x|\to\infty} u(x) = 0\} \\ C_c(\mathbb{R}^n) &= \{u \in C_b(\mathbb{R}^n) : (\exists\, K > 0) \ni (\forall\, x \text{ with } |x| \geq K)\ u(x) = 0\} \end{aligned}$$

• Suppose M is a metric space and $u : M \to \mathbb{F}$ is a function. The *support* of u is the *closure* of $\{x \in M : u(x) \neq 0\}$. The complement of the support of a function is the interior of $\{x \in M : u(x) = 0\}$.

• Elements of $C_c(\mathbb{R}^n)$ are continuous functions with *compact support*.

• $C_0(\mathbb{R}^n)$ is complete in the sup-norm. This can either be shown directly, or by showing that $C_0(\mathbb{R}^n)$ is a closed subspace of $C_b(\mathbb{R}^n)$.

• $C_c(\mathbb{R}^n)$ is *not* complete. In fact, $C_c(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$. So $C_0(\mathbb{R}^n)$ is a representation of the completion of $C_c(\mathbb{R}^n)$ in the sup-norm.

Series in Normed Linear Spaces

Let $(V, \|\cdot\|)$ be a normed linear space. Consider a series $\sum_{n=1}^{\infty} v_n$ in V.

Definition. We say the series *converges in* V if
$\exists\, v \in V$ such that $\lim_{N\to\infty} \|S_N - v\| = 0$, where $S_N = \sum_{n=1}^{N} v_n$ is the N^{th} partial sum. We say this series *converges absolutely* if $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Caution: Strictly speaking, if a series "converges absolutely" in a normed linear space, it does not have to converge in that space. For example, the series $(1, 0\cdots) + \left(0, \frac{1}{2}, 0\cdots\right) + \left(0, 0, \frac{1}{4}, 0\cdots\right)$ "converges absolutely" in $\mathbb{F}_0^{\infty}$, but it doesn't converge in $\mathbb{F}_0^{\infty}$.

Proposition. A normed linear space $(V, \|\cdot\|)$ is complete iff every absolutely convergent series converges in $(V, \|\cdot\|)$.

Proof Sketch
($\Rightarrow$) Given an absolutely convergent series, show that the sequence of partial sums is Cauchy: for $m > n$, $\|S_m - S_n\| \le \sum_{j=n+1}^{m} \|v_j\|$.
($\Leftarrow$) Given a Cauchy sequence $\{x_n\}$, choose $n_1, n_2 < \cdots$ inductively so that for $k = 1, 2, \ldots,$ $(\forall\, n, m \ge n_k)$ $\|x_n - x_m\| \le 2^{-k}$. Then $\|x_{n_k} - x_{n_{k+1}}\| \le 2^{-k}$. The series $x_{n_1} + \sum_{k=2}^{\infty}(x_{n_k} - x_{n_{k-1}})$ is absolutely convergent. Let x be its limit. Then $x_n \to x$.