Linear Analysis
Lecture 5

Inner Products and V'

Let $\dim V < \infty$ with inner product $\langle \cdot, \cdot \rangle$.
Choose a basis $\mathcal{B}$ and let $v, w \in V$ have coordinates in $\mathbb{F}^n$ given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \text{respectively.}$$

Let $A \in \mathbb{F}^{n \times n}$ be the inner product matrix in this basis, then

$$w^*(v) = \langle v, w \rangle = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \overline{y_j} \right) x_i.$$

It follows that w^* has components

$$b_i = \sum_{j=1}^{n} a_{ij} \overline{y_j}$$

with respect to the dual basis.
Therefore, the map $w \mapsto w^*$ corresponds to a mapping of its coordinates in the basis $\mathcal{B}$ to its coordinates in the dual basis $\mathcal{B}'$ given by the matrix-vector product

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = A \overline{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}.$$

Annihilators and Orthogonal Projections

Suppose $W \subset V$ is a subspace and define the orthogonal complement (read W "perp")

$$W^{\perp} = \{v \in V : \langle v, w \rangle = 0 \quad (\forall\, w \in W)\}.$$

The orthogonal complement $W^{\perp}$ is a subspace of V.

The use of the same notation as used for the annihilator of W is justified since the image of $W^{\perp}$ under the map $w \to w^{*}$ is precisely the annihilator of W.

If $\dim V < \infty$, a dimension count and the obvious $W \cap W^{\perp} = \{0\}$ show that

$$V = W \oplus W^{\perp}\ .$$

So in a finite dimensional inner product space, a subspace W determines a natural complement, namely $W^{\perp}$.

The induced projection onto W (along $W^{\perp}$) is called the *orthogonal projection* onto W.

Norms

A *norm* on a vector space V is a function $\|\cdot\| : V \to [0, \infty)$ satisfying

(i) $(\forall\, v \in V) \quad \|v\| \geq 0$, and $\|v\| = 0$ iff $v = 0$

(ii) $(\forall\, \alpha \in \mathbb{F})(\forall\, v \in V) \quad \|\alpha v\| = |\alpha| \cdot \|v\|$, and

(iii) (triangle inequality) $(\forall\, v, w \in V) \quad \|v + w\| \leq \|v\| + \|w\|$.

The pair $(V, \|\cdot\|)$ is called a *normed linear space* (or normed vector space).

Fact: A norm $\|\cdot\|$ on a vector space V induces a metric d on V by

$$d(v, w) = \|v - w\|.$$

(1) ℓ^p-norm on $\mathbb{F}^n$ $(1 \le p \le \infty)$

(a) $p = \infty$: $\|x\|_\infty = \max_{1 \le i \le n} |x_i|, \ x \in \mathbb{F}^n$

(b) $1 \le p < \infty$: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}, \ x \in \mathbb{F}^n.$

The triangle inequality

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

is known as "Minkowski's inequality." It is a consequence of Hölder's inequality.

Integral versions of these inequalities are proved in real analysis texts, e.g., Folland, Royden or Rudin. The proofs for vectors in $\mathbb{F}^n$ are analogous to the proofs for integrals

(2) ℓ^p-norm on ℓ^p (subspace of $\mathbb{F}^\infty$) $(1 \leq p \leq \infty)$

(a) $p = \infty$:

$$\ell^\infty = \{x \in \mathbb{F}^\infty : \sup_{i \geq 1} |x_i| < \infty\} \quad \|x\|_\infty = \sup_{i \geq 1} |x_i|$$

for $x \in \ell^\infty$.

(b) $1 \leq p < \infty$:

$$\ell^p = \left\{ x \in \mathbb{F}^\infty : \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} < \infty \right\}, \quad \|x\|_p = \left(\sum_{i=1}^{\infty} |x|^p \right)^{\frac{1}{p}}$$

for $x \in \ell^p$.

Examples of Normed Linear Spaces

(3) L^p norm on $C([a,b]) \quad (1 \le p \le \infty)$

(a) $p = \infty$: $\quad \|f\|_\infty = \sup_{a \le x \le b} |f(x)|$. Since $|f(x)|$ is a continuous, real-valued function on the compact set $[a,b]$, it takes on its maximum, so the "sup" is actually a "max" here:

$$\|f\|_\infty = \max_{a \le x \le b} |f(x)|.$$

(b) $1 \le p < \infty$: $\quad \|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$. Use continuity of f to show that

$$\|f\|_p = 0 \Rightarrow f(x) \equiv 0 \quad \text{on} \quad [a,b].$$

The triangle inequality

$$\left(\int_a^b |f(x) + g(x)|^p \, dx \right)^{\frac{1}{p}} \le \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p \, dx \right)^{\frac{1}{p}}$$

is Minkowski's inequality, a consequence of Hölder's inequality:

$$\int_a^b f(x) g(x) dx \ \le \ \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}} ,$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 \ .$$

Theorem: $(V,\ \|\cdot\|_v)$ and $(W,\ \|\cdot\|_w)$ are normed linear spaces. $L : V \to W$ is a linear transformation. Then the following are equivalent:

(a) L is continuous

(b) L is uniformly continuous (*Lipschitz continuous*)

(c) $(\exists\, C)$ so that $(\forall\, v \in V)\quad \|Lv\|_w \le C\|v\|_v$.

Proof: (a) $\Rightarrow$ (c): Suppose L is continuous. Then L is continuous at $v = 0$. Let $\epsilon = 1$. Then $\exists\, \delta > 0$ such that

$$\|v\|_v \le \delta \quad \Rightarrow \quad \|Lv\|_w \le 1$$

(as $L(0) = 0$). For any $v \ne 0$,

$$\|\,\delta v/\|v\|_v\,\|_v \le \delta \quad \Rightarrow \quad \|L\,(\delta v/\|v\|_v)\|_w \le 1,$$

i.e., $\|Lv\|_w \le \frac{1}{\delta}\|v\|_v$. Let $C = \frac{1}{\delta}$.
(c) $\Rightarrow$ (b): Condition (c) implies that

$$(\forall\, v_1, v_2 \in V)\quad \|Lv_1 - Lv_2\|_w = \|L(v_1 - v_2)\|_w \le C\|v_1 - v_2\|_v.$$

Hence L is uniformly continuous (given ϵ, let $\delta = \frac{\epsilon}{c}$, etc.). In fact, L is uniformly Lipschitz continuous with Lipschitz constant C.
(b)$\Rightarrow$(a) is immediate. □

Bounded Linear Operators and Their Norms

Definition: V and W are normed linear spaces. $L : V \to W$ a linear operator. If

$$\|L\| := \sup_{v \in V, v \neq 0} \frac{\|Lv\|_w}{\|v\|_v} < \infty \ ,$$

then L is called a *bounded linear operator* from V to W, in which case we call $\|L\|$ the *operator norm* of L.

Remarks.

(1) Note that it is the *norm ratio* $\frac{\|Lv\|_w}{\|v\|_v}$ (or "stretching factor") that is bounded, *not* $\{\|Lv\|_w : v \in V\}$.

(2) The theorem above says that if V and W are normed linear spaces and $L : V \to W$ is linear, then

L is continuous $\Leftrightarrow$ L is uniformly continuous $\Leftrightarrow$ L is a bounded linear operator.

Equivalence of Norms

It is easily seen that the norm in a normed linear space is a continuous mapping from the space into $\mathbb{R}$. This follows from the *other half* of the triangle inequality:

$$\left| \, \|u\| - \|v\| \, \right| \leq \| \, u - v \, \| \; .$$

Definition: Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$, both on the same vector space V, are called *equivalent norms* on V if $\exists$ constants $C_1, C_2 > 0$ for which

$$(\forall \, v \in V) \quad C_1 \|v\|_2 \leq \|v\|_1 \leq C_2 \|v\|_2 \; .$$

Fact: V is finite-dim if and only if any two norms on V are equivalent.

Remarks.

(1) All norms on a fixed finite dimensional vector space are equivalent. However, the constants C_1 and C_2 can depend on the dimension. For example, in $\mathbb{F}^n$

$$\|x\|_2 \leq \sqrt{n} \|x\|_\infty \; .$$

(2) In a normed linear space V, the closed unit ball $\mathbb{B} := \{v \in V : \|v\| \leq 1\}$ is compact iff $\dim V < \infty$.

Examples

(1) Set $$\mathbb{F}_0^\infty = \{x \in \mathbb{F}^\infty : (\exists\, N)(\forall\, n \geq N) \quad x_n = 0\}.$$

For $1 \leq p < q \leq \infty$, the ℓ^p and ℓ^q norms are *not* equivalent.
Consider $p = 1,\ q = \infty$. Note that
$$\|x\|_\infty \leq \sum_{i=1}^{\infty} |x_i| = \|x\|_1\ .$$
But if
$$y_1 = (1, 0, 0 \cdots),\ y_2 = (1, 1, 0, \cdots),\ y_3 = (1, 1, 1, 0, \cdots),\ \ldots$$
then $$\|y_n\|_\infty = 1 \quad \text{and} \quad \|y_n\|_1 = n \quad \forall\, n\ .$$

So there does *not* exist a constant C for which
$$(\forall\, x \in \mathbb{F}_0^\infty) \qquad \|x\|_1 \leq C\|x\|_\infty\ .$$

(2) In ℓ^2 (a subspace of $\mathbb{F}^\infty$) with norm $\|x\|_2 = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$, the closed unit ball $\{x \in \ell^2 : \|x\|_2 \leq 1\}$ is *not* compact. The sequence $e_1, e_2, e_3, \ldots$ is bounded, $\|e_i\|_{\ell^2} \leq 1$, and all are in the closed unit ball, but no subsequence can converge because
$\|e_i - e_j\|_{\ell^2} = \sqrt{2} \quad \text{for} \quad i \neq j\ .$

Norms induced by inner products

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Define $\|v\| = \sqrt{\langle v, v \rangle}$. We have

$$\|v\| \geq 0 \quad \text{with} \quad \|v\| = 0 \;\Leftrightarrow\; v = 0 \,, \quad \text{and}$$
$$(\forall\, \alpha \in \mathbb{F})(\forall\, v \in V) \qquad \|\alpha v\| = |\alpha| \cdot \|v\| \,.$$

To show that $\| \cdot \|$ is a norm we need the triangle inequality.

Note that for any two vectors $u, v \in V$ we have

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v,\, u + v \rangle \\
&= \langle u,\, u \rangle + \langle u,\, v \rangle + \langle v,\, u \rangle + \langle v,\, v \rangle \\
&= \langle u,\, u \rangle + \langle u,\, v \rangle + \overline{\langle u,\, v \rangle} + \langle v,\, v \rangle \\
&= \|u\|^2 + 2\mathcal{R}e\, \langle u,\, v \rangle + \|v\|^2 \,.
\end{aligned}$$

Consequently, if u and v are orthogonal ($\langle u,\, v \rangle = 0$), then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 \,.$$

The Cauchy-Schwarz Inequality

For all $v, w \in V$ we have

$$|\langle v, w\rangle| \leq \|v\| \cdot \|w\| \ ,$$

with equality iff v and w are linearly dependent.

Proof: Case (i) If $v = 0$, we are done.

Case (ii) Assume $v \neq 0$, and set

$$u := w - \frac{\langle w,\ v\rangle}{\|v\|^2} v,$$

so that $\langle u,\ v\rangle = 0$, i.e. $u \perp v$. Then, by orthogonality,

$$\|w\|^2 = \left\| u + \frac{\langle w,\ v\rangle}{\|v\|^2} v \right\|^2 = \|u\|^2 + \left\| \frac{\langle w,\ v\rangle}{\|v\|^2} v \right\|^2 = \|u\|^2 + \frac{|\langle w,\ v\rangle|^2}{\|v\|^2} \geq \frac{|\langle w,\ v\rangle|^2}{\|v\|^2},$$

with equality iff $u = 0$. $\square$

The Triangle Inequality

For all $v, w \in V$ we have

$$\begin{aligned} \|v+w\|^2 &= \langle v+w, v+w \rangle \\ &= \langle v, v \rangle + 2\mathcal{R}e\langle v, w \rangle + \langle w, w \rangle \\ &\leq \|v\|^2 + 2|\langle v, w \rangle| + \|w\|^2 \\ &\leq \|v\|^2 + 2\|v\| \cdot \|w\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2. \end{aligned}$$

So $\|v\| = \sqrt{\langle v, v \rangle}$ is a norm on V.

The resulting norm is called the norm induced by the inner product $\langle \cdot, \cdot \rangle$. That is, an inner product induces a norm which, in turn, induces a metric

$$(V, \langle \cdot, \cdot \rangle) \leftrightarrow (V, \| \cdot \|) \leftrightarrow (V, d) \ .$$

Examples

(1) The Euclidean norm [i.e. ℓ^2 norm] on $\mathbb{F}^n$ is induced by the standard inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} \quad : \quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i \overline{x_i}} = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

(2) Let $A \in \mathbb{F}^{n \times n}$ be Hermitian-symmetric and positive definite, and let

$$\langle x, y \rangle_A = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} \overline{y_j} \quad \text{for } x, y \in \mathbb{F}^n.$$

Then $\langle \cdot, \cdot \rangle_A$ is an inner product on $\mathbb{F}^n$, which induces the norm

$$\|x\|_A = \sqrt{\langle x, x \rangle_A} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} \overline{x_j}} = \sqrt{x^T A \overline{x}} = \sqrt{x^H \overline{A} x}.$$

Examples

(3) The ℓ^2-norm on ℓ^2 (subspace of $\mathbb{F}^\infty$) is induced by the inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} \quad : \quad \|x\|_2 = \sqrt{\sum_{i=1}^{\infty} x_i \overline{x_i}} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}.$$

(4) The L^2 norm

$$\|u\|_2 = \left(\int_a^b |u(x)|^2 \, dx \right)^{\frac{1}{2}}$$

on $C([a, b])$ is induced by the inner product

$$\langle u, v\rangle = \int_a^b u(x)\overline{v(x)}\, dx \ .$$