# Linear Analysis
# Lecture 4

# Dual Transformations

Suppose $U$, $V$ and $W$ are vector spaces, possibly infinite dim.

$L \in \mathcal{L}(V, W)$ the space of all linear transformations from $V$ to $W$.

Define the **dual**, or **adjoint** transformation $L^* : W' \to V'$ by

$$(L^* g)(v) = g(Lv) \quad \text{for} \quad g \in W',\ v \in V.$$

$L \mapsto L^*$ is a linear transformation from $\mathcal{L}(V, W)$ to $\mathcal{L}(W', V')$, and

$$(L \circ M)^* \ = \ M^* \circ L^* \quad \text{if} \quad M \in \mathcal{L}(U, V).$$

since

$$\begin{aligned}
((L \circ M)^* g)(u) &= g((L \circ M)u) = g(L(Mu)) \\
&= L^*(g)(Mu) = M^*(L^* g)(u) \\
&= (M^* \circ L^*)(g)(u)\ .
\end{aligned}$$

# Matrices for Dual Transformations

Suppose $V$ and $W$ are finite dimensional. Let bases for $V$ and $W$ be chosen along with corresponding dual bases for $V^*$ and $W^*$.

Let $L \in \mathcal{L}(V, W)$ have the matrix representation $T$ in the given bases. Let $v \in V$ and $Lv \in W$ have coordinates

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad \text{respectively,} \quad \text{so} \quad y = Tx.$$

If $g$ has coordinates $b = (b_1 \cdots b_m)$, then

$$g(Lv) = (b_1 \cdots b_m)\, T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

so $L^* g$ has coordinates

$$(a_1 \cdots a_n) = (b_1 \cdots b_m)\, T \quad \text{or} \quad a = bT\ .$$

# Matrices for Dual Transformations

Thus, $L$ is represented by left-multiplication by $T$ on column vectors, and $L^*$ is represented by right-multiplication by $T$ on row vectors.

Using the obvious isometry, one can also represent the dual coordinate vectors also as column vectors.

Taking the transpose in

$$(a_1 \cdots a_n) = (b_1 \cdots b_m)\, T.$$

gives

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = T^{\mathrm{T}} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

That is $L^*$ can also be represented through left-multiplication by $T^{\mathrm{T}}$ on column vectors. ($T^{\mathrm{T}}$ is the transpose of $T$: $(T^{\mathrm{T}})_{ij} = t_{ji}$.)

# The Double Algebraic Dual

The algebraic dual of $V'$ is $V''$. There is a natural inclusion $V \to V''$. If $v \in V$, then $f \mapsto f(v)$ defines a linear functional on $V'$. This map is injective (one to one). Indeed, if $v \neq 0$, there is an $f \in V'$ for which $f(v) \neq 0$.

We identify $V$ with its image, so we can regard $V \subset V''$.

If $V$ is finite dimensional, then $V = V''$ since

$$\dim V = \dim V' = \dim V''.$$

If $V$ is infinite dimensional, however, then there may be elements of $V''$ which are not in $V$.

# Annihilators

Let $S \subset V$. Define the annihilator $S^\perp \subset V'$ by

$$S^\perp = \{f \in V' : (\forall\, v \in S)\ f(v) = 0\}.$$

Clearly $S^\perp = (\mathrm{Span}(S))^\perp$ , and $S^{\perp\perp} \subset V''$.

**Proposition.** If $\dim V < \infty$, then $S^{\perp\perp} = \mathrm{Span}(S)$.

**Proof**: As above we make the identification $V = V''$ and so

$$\mathrm{Span}(S) \subset S^{\perp\perp}.$$

For the reverse, we assume WLOG that $S$ is a subspace with basis $\{s_1, \dots, s_m\}$ which we complete to the basis $\{s_1, \dots, s_{m+1}, \dots, s_n\}$ of $V$. Then the dual basis vectors $\{f_{m+1}, \dots, f_n\}$ are a basis for $S^\perp$. So

$$\dim S^{\perp\perp} = n - \dim S^\perp = n - (n - \dim S) = \dim S\ .$$

Since $S \subset S^{\perp\perp}$, the proof is complete.

# The Fundamental Theorem of the Alternative

**Proposition.** Suppose $L \in \mathcal{L}(V, W)$. Then $\mathcal{N}(L^*) = \mathcal{R}(L)^\perp$.

**Proof**: Clearly both are subspaces of $W'$. Let $g \in W'$. Then

$$\begin{aligned} g \in \mathcal{N}(L^*) &\iff L^* g = 0 \\ &\iff (\forall v \in V)\ (L^* g)(v) = 0 \\ &\iff (\forall\, v \in V)\ g(Lv) = 0 \\ &\iff g \in \mathcal{R}(L)^\perp\ . \end{aligned} \qquad \square$$

The result is called the *Fundamental Theorem of the Alternative* since it is equivalent to the following:
One of the two alternatives (A) and (B) must hold, and both (A) and (B) cannot hold.

$$(A) \qquad \left[ \begin{array}{c} \text{The system} \\ y = Lx \\ \text{is solvable.} \end{array} \right]$$

$$(B) \qquad \left[ \begin{array}{c} \text{There exists } w \in W' \text{ such that} \\ L^* w = 0 \\ \text{and } w(y) \neq 0. \end{array} \right]$$

# Bilinear Forms

A function $\varphi : V \times V \to \mathbb{F}$ is called a **bilinear form** if it is linear in each variable separately:

$$\varphi(\sum x_i v_i, \ \sum y_j v_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \varphi(v_i, v_j) \ .$$

**Examples**:

**(1)** For $A \in \mathbb{F}^{n\times n}$, the function

$$\varphi(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j$$

is a bilinear form. In fact, all bilinear forms on $\mathbb{F}^n$ are of this form, as

$$\varphi(\sum x_i e_i \sum y_j e_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \varphi(e_i, e_j), \quad \text{so just set } a_{ij} = \varphi(e_i, e_j).$$

In general, let $V$ be finite-dimensional with basis $\{v_1, \dots, v_n\}$. Let $v, w \in V$ with $v = \sum x_i v_i$ and $w = \sum y_j v_j$. If $\varphi$ is bilinear on $V$, then

$$\varphi(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \varphi(v_i, v_j) = x^T A y,$$

where $A \in \mathbb{F}^{n\times n}$ satisfies $a_{ij} = \varphi(v_i, v_j)$.
$A$ is called the **matrix of** $\varphi$ with respect to the basis $\{v_1, \dots, v_n\}$.

# Bilinear Forms: Examples

(2) One can also use infinite matrices $(a_{ij})_{i,j\geq 1}$ for $V = \mathbb{F}^\infty$ as long as convergence conditions are imposed. For example, if all $|a_{ij}| \leq M$, then

$$\varphi(x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}x_i y_j$$

defines a bilinear form on $\ell^1$ since

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}x_i y_j| \leq M(\sum_{i=1}^{\infty} |x_i|)(\sum_{j=1}^{\infty} |y_j|).$$

(3) If $f, g \in V'$, then $\varphi(x, y) = f(x)g(y)$ is a bilinear form.

(4) If $V = C[a, b]$, then

**(i)** for $k \in C([a, b] \times [a, b])$, $\int_a^b \int_a^b k(x, y)u(x)v(y)\,dxdy$

**(ii)** for $h \in C([a, b])$, $\int_a^b h(x)u(x)v(x)\,dx$

**(iii)** for $x_0 \in [a, b]$, $u(x_0) \int_a^b v(x)\,dx$

are all examples of bilinear forms.

# Symmetric Bilinear Forms

A bilinear form is **symmetric** if

$$(\forall\, v, w \in V) \quad \varphi(v, w) = \varphi(w, v).$$

In the finite-dimensional case, this implies the matrix $A$ be symmetric (wrt any basis), i.e.,

$$A = A^{\mathrm{T}}, \quad \text{or} \quad (\forall\, i, j)\ a_{ij} = a_{ji}.$$

# Sesquilinear Forms

Let $\mathbb{F} = \mathbb{C}$, a sesquilinear form on $V$, $\varphi : V \times V \to \mathbb{C}$, is linear in the first variable and conjugate-linear in the second variable, i.e.,

$$\varphi(v, \alpha_1 w_1 + \alpha_2 w_2) = \bar{\alpha}_1 \varphi(v_1, w_1) + \bar{\alpha}_2 \varphi(v, w_2).$$

On $\mathbb{C}^n$ all sesquilinear forms are of the form

$$\varphi(z, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i \bar{w}_j \qquad \text{for some } A \in \mathbb{C}^{n \times n}.$$

To be able to discuss bilinear forms over $\mathbb{R}$ and sesquilinear forms over $\mathbb{C}$ at the same time, we will speak of a sesquilinear form over $\mathbb{R}$ and mean just a bilinear form over $\mathbb{R}$.

A sesquilinear form is said to be *Hermitian-symmetric* (or Hermitian) if

$$(\forall\, v, w \in V) \quad \varphi(v, w) = \overline{\varphi(w, v)}$$

(when $\mathbb{F} = \mathbb{R}$, we say the form is symmetric). This corresponds to the condition that $A = A^{\mathrm{H}}$ where $A^{\mathrm{H}} = \bar{A}^{\mathrm{T}}$ i.e., $(A^{\mathrm{H}})_{ij} = \overline{A_{ji}}$ .

$A^{\mathrm{H}}$ is the Hermitian transpose (or conjugate transpose) of $A$ when $\mathbb{F} = \mathbb{C}$.

If $A = A^{\mathrm{H}} \in \mathbb{C}^{n \times n}$, we say $A$ is **Hermitian (-symmetric)**.
If $A = A^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ and $\mathbb{F} = \mathbb{R}$, we say $A$ is symmetric.

# Inner Products

Associate the quadratic form $\varphi(v, v)$ with the sesquilinear form $\varphi$.

We say that $\varphi$ is nonnegative (or positive semi-definite) if

$$(\forall\, v \in V) \quad \varphi(v, v) \geq 0,$$

and that $\varphi$ is positive (or positive definite) if

$$(\forall\, v \in V, v \neq 0) \quad \varphi(v, v) > 0.$$

By an **inner product** on $V$, we will mean a positive-definite Hermitian-symmetric sesquilinear form.

**inner product**

$=$

**positive-definite Hermitian-symmetric sesquilinear form**

# Examples of Inner Products

$\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$

**(1)** $\mathbb{F}^n$ with the Euclidean inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} = y^{\mathrm{H}} x.$$

**(2)** Let $V = \mathbb{F}^n$ and $A \in \mathbb{F}^{n \times n}$ be Hermitian-symmetric and define

$$\langle x, y \rangle_A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i \overline{y_j} = y^{\mathrm{H}} A x.$$

The requirement that $\langle x, x \rangle_A > 0$ for $x \neq 0$ so that $\langle \cdot, \cdot \rangle_A$ is an inner product serves to define positive-definite matrices.

# Examples of Inner Products

**(3)** Let $V$ be any finite-dimensional vector space. Choose a basis and thus identify $V \cong \mathbb{F}^n$. Transfer the Euclidean inner product to $V$ in the coordinate of this basis. The resulting inner product depends on the choice of basis.
With respect to the coordinates induced by a basis, any inner product on a finite-dimensional vector space $V$ is of the form described in example (2) above.

**(4)** One can define an inner product on $\ell^2$ by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

To see that this sum converges absolutely, apply the finite-dimensional Cauchy-Schwarz inequality to obtain

$$\sum_{i=1}^{n} |x_i \overline{y_i}| \leq \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |y_i|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |y_i|^2 \right)^{\frac{1}{2}}.$$

Let $n \to \infty$ to see that the series $\sum_{i=1}^{\infty} x_i \overline{y_i}$ converges absolutely.

**(5)** The $L^2$-inner product on $C([a, b])$ is given by

$$\langle u, v \rangle = \int_a^b u(x) \overline{v(x)} dx.$$

# Inner Products and $V'$

An inner product on $V$ determines an injection $V \to V'$.
For $w \in V$, define

$$w \mapsto w^* \in V' \quad \text{by} \quad w^*(v) = \langle v, w \rangle.$$

Since $w^*(w) = \langle w, w \rangle$ it follows that

$$w^* = 0 \Rightarrow w = 0 \ ,$$

so the map $w \mapsto w^*$ is injective (one to one).

- The map $w \mapsto w^*$ is **conjugate-linear** since

$$(\alpha w)^* = \bar{\alpha} w^* \ .$$

It is linear if $\mathbb{F} = \mathbb{R}$.

- The image of the mapping $w \mapsto w^*$ is a subspace of $V'$.

If $\dim V < \infty$, then this map is surjective too since $\dim V = \dim V'$.

In general, the mapping $w \mapsto w^*$ is not surjective.

# Inner Products and $V'$

Let $\dim V < \infty$ with inner product $\langle \cdot, \cdot \rangle$.
Choose a basis $\mathcal{B}$ and let $v, w \in V$ have coordinates in $\mathbb{F}^n$ given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \text{respectively.}$$

Let $A \in \mathbb{F}^{n \times n}$ be the inner product matrix in this basis, then

$$w^*(v) = \langle v, w \rangle = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} \overline{y_j} \right) x_i.$$

It follows that $w^*$ has components

$$b_i = \sum_{j-1}^{n} a_{ij} \overline{y_j}$$

with respect to the dual basis.
Therefore, the map $w \mapsto w^*$ corresponds to a mapping of its coordinates in the basis $\mathcal{B}$ to its coordinates in the dual basis $\mathcal{B}'$ given by the matrix-vector product

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = A \overline{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}.$$