Linear Analysis
Lecture 24

Fundamental Estimate

Let $f(t, x)$ be continuous in t and x, and uniformly Lipschitz continuous in x with Lipschitz constant L. Consider the DE

$$(*) \qquad\qquad x' = f(t, x) \ .$$

Let $\epsilon_1, \epsilon_2 > 0$, and suppose

$x_i(t)$ is an ϵ_i-approximate solution of (*) on I, $i = 1, 2.$.

Given $t_0 \in I$, suppose that

$$|x_1(t_0) - x_2(t_0)| \leq \delta \ .$$

Then, for $t \in I$,

$$|x_1(t) - x_2(t)| \leq \delta e^{L|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{L}(e^{L|t-t_0|} - 1) \ .$$

Corollary: Uniform Convergence of Solutions to IVPs

Corollary: For $j \geq 1$, suppose $x_j(t)$ is a solution to the DE

$$x_j' = f_j(t, x_j) \text{ on } I := [a, b],$$

and suppose $x(t)$ is a solution to the DE

$$x' = f(t, x) \text{ on } I,$$

where the f_j's and f are continuous in t and x and f is Lipschitz in x.

If $f_j \to f$ uniformly on $[a, b] \times \mathbb{F}^n$ and there is a $t_0 \in I$ such that $x_j(t_0) \to x(t_0)$, then

$$x_j(t) \to x(t) \text{ uniformly on } [a, b].$$

Proof: By the Fundamental Estimate, with $x_1 = x_j$ and $x_2 = x$,

$$|x_j(t) - x(t)| \leq |x_j(t_0) - x(t_0)| e^{L|t-t_0|} + \frac{\epsilon_j}{L}(e^{L|t-t_0|} - 1),$$

where $\epsilon_j := \sup_{t \in [a,b]} |f_j(t, x_j(t)) - f(t, x_j(t))|$.

Parameterized Differential Equations

Consider a family of IVPs

$$x' = f(t, x, \mu), \quad x(t_0) = y,$$

where $\mu \in \mathbb{F}^m$ is a vector of parameters and $y \in \mathbb{F}^n$. Assume for each value of μ, $f(t, x, \mu)$ is continuous in t and x and Lipschitz in x with Lipschitz constant L locally independent of μ. For each fixed $(\mu, y) \in \mathbb{F}^m \times \mathbb{F}^n$, this is a standard IVP having a solution $x(t, \mu, y)$.

Theorem. If f is continuous in (t, x, μ) and Lipschitz in x with Lipschitz constant independent of t and μ, then $x(t, \mu, y)$ is jointly continuous in (t, μ, y).

Proof: The previous corollary implies that x is continuous in μ, y, uniformly in t. Since each $x(t, \mu, y)$ is continuous in t for given μ, y, we can restate this result as saying the mapping $(\mu, y) \mapsto x(t, \mu, y)$ from $\mathbb{F}^m \times \mathbb{F}^n$ into $(C[a, b], \|\cdot\|_\infty)$ is continuous. Standard arguments now show x is continuous in (t, μ, y) jointly.

Transforming Initial Conditions into Parameters

Suppose $f(t, x)$ maps an open subset $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$ into $\mathbb{F}^n$, where f is continuous in t and x and locally Lipschitz in x on $\mathcal{D}$.

Let $(\tau, y) \in \mathcal{D}$, and consider the IVP

$$x' = f(t, x), \;\; x(\tau) = y.$$

Think of τ as a *variable initial time* t_0, and y as a *variable initial value* x_0. Viewing (τ, y) as parameters, let $x(t, \tau, y)$ be the solution of this IVP.

One can show that if $(t_0, x_0) \in \mathcal{D}$ and $x(t, t_0, x_0)$ exists in $\mathcal{D}$ on a time interval $[t_0, t_0 + b]$, then for (τ, y) in a sufficiently small open neighborhood $\mathcal{N}$ of (t_0, x_0), the solution $x(t, \tau, y)$ exists on

$$I_{\tau, t_0} \equiv [\min(\tau, t_0), \max(\tau, t_0) + b]$$

(which contains $[t_0, t_0 + b]$ and $[\tau, \tau + b]$), and moreover $\{(t, x(t, \tau, y)) : t \in I_{\tau, t_0}, (\tau, y) \in \mathcal{N}\}$ is contained in some compact subset of $\mathcal{D}$.

Transforming Initial Conditions into Parameters

Define

$$\widetilde{f}(t, x, \tau, y) = f(\tau + t, x + y) \quad \text{and} \quad \widetilde{x}(t, \tau, y) = x(\tau + t, \tau, y) - y.$$

Then $\widetilde{x}(t, \tau, y)$ is a solution of the IVP

$$\widetilde{x}' = \widetilde{f}(t, \widetilde{x}, \tau, y), \quad \widetilde{x}(0) = 0$$

with $n + 1$ parameters (τ, y) and fixed initial conditions. This IVP is equivalent to the original IVP

$$x' = f(t, x), \; x(\tau) = y.$$

Remarks

(1) $\widetilde{f}$ is continuous in t, x, τ, y and locally Lipschitz in x in the open set

$$\mathcal{W} \equiv \{(t, x, \tau, y) : (\tau + t, x + y) \in \mathcal{D}, (\tau, y) \in \mathcal{D}\} \subset \mathbb{R} \times \mathbb{F}^n \times \mathbb{R} \times \mathbb{F}^n.$$

(2) If f is C^k in t, x in $\mathcal{D}$, then $\widetilde{f}$ is C^k in t, x, τ, y in .

Transforming Parameters into Initial Conditions

Suppose $f(t, x, \mu)$ is continuous in t, x, μ and locally Lipschitz in x on an open set $\subset \mathbb{R} \times \mathbb{F}^n \times \mathbb{F}^m$. Consider the IVP $x' = f(t, x, \mu)$, $x(t_0) = x_0$, with solution $x(t, \mu)$. Introduce a new variable $z \in \mathbb{F}^m$ which we think of as the solution to the IVP $z' = 0$, $z(t_0) = \mu$, so that $z(t) \equiv \mu$. Define

$$\widetilde{x} = \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{F}^{n+m} \quad \text{and} \quad \widetilde{f}(t, \widetilde{x}) = \begin{bmatrix} f(t, x, z) \\ 0 \end{bmatrix}.$$

Consider the IVP

$$\widetilde{x}' = \widetilde{f}(t, \widetilde{x}), \quad \widetilde{x}(t_0) = \begin{bmatrix} x_0 \\ \mu \end{bmatrix},$$

(i.e., $x' = f(t, x, z)$, $z' = 0$, $x(t_0) = x_0$, $z(t_0) = \mu$), with solution $\widetilde{x}(t, \mu)$. Then

$$\widetilde{x}(t, \mu) = \begin{bmatrix} x(t, \mu) \\ \mu \end{bmatrix},$$

and the two IVPs are equivalent.

Transforming Parameters into Initial Conditions: Remarks

Remarks

(1) If f is continuous in (t, x, μ) and locally Lipschitz in (x, μ), then $\widetilde{f}$ is continuous in $t, \widetilde{x}$ and locally Lipschitz in $\widetilde{x}$.

(2) If f is C^k in t, x, μ, then $\widetilde{f}$ is C^k in $t, \widetilde{x}$.

(3) One can show that if $(t_0, x_0, \mu_0) \in \mathcal{W}$ and the solution $x(t, \mu_0)$ exists in $\mathcal{W}$ on a time interval $[t_0, t_0 + b]$, then for μ in some sufficiently small open neighborhood $\mathcal{U}$ of μ_0 in $\mathbb{F}^m$, the solution $x(t, \mu)$ exists on $[t_0, t_0 + b]$, and, moreover, the set

$$\{(t, x(t, \mu), \mu) : t \in [t_0, t_0 + b], \mu \in \mathcal{U}\}$$

is contained in some compact subset of $\mathcal{W}$.

(4) An IVP $x' = f(t, x, \mu)$, $x(\tau) = y$ with parameters $\mu \in \mathbb{F}^m$ and variable initial values $\tau \in \mathbb{R}$, $y \in \mathbb{F}^n$ can be transformed similarly into either IVPs with variable IC and no parameters in the DE or IVPs with fixed IC and variable parameters in the DE.