Linear Analysis
Lecture 23

Continuation of Solutions in Time

We consider two kinds of results
(1) *local continuation* (no Lipschitz condition on f)
(2) *global continuation* (for locally Lipschitz f)

Local Continuation (Continuation at a Point)

Assume $x(t)$ is a solution of the DE $x' = f(t, x)$ on an interval I and f is continuous on a subset $\mathcal{S} \subset \mathbb{R} \times \mathbb{F}^n$ containing $\{(t, x(t)) : t \in I\}$.

Case 1: I is closed at the right end, i.e., $I = (-\infty, b]$, $[a, b]$, or $(a, b]$. Assume further that $(b, x(b))$ is in the interior of $\mathcal{S}$. Then the solution can be extended (by Cauchy-Peano) to an interval with right end $b + \beta$ for some $\beta > 0$. This is done by solving the IVP

$$x' = f(t, x) \quad \text{with initial value } x(b) \text{ at } t = b$$

on an interval $[b, b + \beta]$. To show that the continuation is C^1 at $t = b$, note that the extended $x(t)$ satisfies the integral equation

$$x(t) = x(b) + \int_b^t f(s, x(s))\, ds \quad \text{on } I \bigcup [b, b + \beta].$$

Note we do not assume Lipschitz continuity.

Continuation of Solutions in Time; Case 2

Case 2: I is open at the right end, i.e., $I = (-\infty, b)$, $[a, b)$, or (a, b) with $b < \infty$. Assume further that $f(t, x(t))$ is **bounded** on $[t_0, b)$ for some $t_0 < b$ with $[t_0, b) \subset I$, say $|f(t, x(t))| \leq M$ on $[t_0, b)$.
In this case the integral equation

$$(*) \qquad x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds$$

holds for $t \in I$. In particular, for $t_0 \leq \tau \leq t < b$,

$$|x(t) - x(\tau)| = \left| \int_{\tau}^{t} f(s, x(s))\, ds \right| \leq \int_{\tau}^{t} |f(s, x(s))|\, ds \leq M|t - \tau|.$$

Thus, for any sequence $t_n \uparrow b$, $\{x(t_n)\}$ is Cauchy. This implies $\lim_{t \to b^-} x(t)$ exists; call it $x(b^-)$. So $x(t)$ has a **continuous** extension from I to $I \cup \{b\}$.

Comments

• If in addition $(b, x(b^-))$ is in $\mathcal{S}$, then $(*)$ holds on $I \cup \{b\}$ as well, so $x(t)$ is a C^1 **solution** of $x' = f(t, x)$ on $I \cup \{b\}$.

• If $(b, x(b^-))$ is in the interior of $\mathcal{S}$, we are back in Case 1 and can extend the solution $x(t)$ beyond $t = b$.

• The assumption that $f(t, x(t))$ is bounded on $[t_0, b)$ can be restated with a slightly different emphasis: for some $t_0 \in I$, $\{(t, x(t)) : t_0 \leq t < b\}$ stays within a subset of $\mathcal{S}$ on which f is bounded. For example, if $\{(t, x(t)) : t_0 \leq t < b\}$ stays within a compact subset of $\mathcal{S}$, this condition is satisfied.

• The technique of Case 1 can be applied to I is closed at the left end.

• The technique of Case 2 can be applied to I is open at the left end.

Global Continuation

Assume $f(t,x)$ is continuous on an open set $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$ and is locally Lipschitz continuous with respect to x on $\mathcal{D}$. Write $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$.

Let $(t_0, x_0) \in \mathcal{D}$ and consider the IVP

$$x' = f(t,x), \quad x(t_0) = x_0.$$

It has been shown that a unique solutions exist on both $[t_0, t_0 + \alpha_+)$ and $(-\alpha_- + t_0, t_0]$, and that this gives a unique solution on $(-\alpha_- + t_0,\ t_0 + \alpha_+)$ for some $\alpha_+,\ \alpha_- > 0$. Set

$$\begin{aligned} T_+ &= \sup\{t > t_0 \ : \ \exists \text{ a solution of IVP on } [t_0, t)\}, \quad \text{and} \\ T_- &= \inf\{t < t_0 \ : \ \exists \text{ a solution of IVP on } (t,\ t_0]\}. \end{aligned}$$

(T_-, T_+) is the maximal interval of existence of the solution of the IVP. It is possible that $T_+ = \infty$ and/or $T_- = -\infty$.

The maximal interval (T_-, T_+) must be open: if the solution could be extended to T_+ (or T_-), this would contradict the local continuation results since $\mathcal{D}$ is open. Ideally, $T_+ = +\infty$ and $T_- = -\infty$.

Global Continuation

Another posibility is if $f(t,x)$ is not defined for $t \geq T_+$. For example, if $a(t) = \frac{1}{1-t}$, and $x'(t) = a(t)$. Here we don't expect the solution to exist beyond $t = 1$.

But less desirable behavior can occur.

For example, for the IVP:

$$x' = x^2, \ x(0) = x_0 > 0, \ t_0 = 0,$$

and $\mathcal{D} = \mathbb{R} \times \mathbb{R}$. The solution $x(t) = (x_0^{-1} - t)^{-1}$ blows up at $T_+ = 1/x_0$ (note that $T_- = -\infty$). Observe that $x(t) \to \infty$ as $t \to (T_+)^-$. So the solution does not just "stop" in the interior of $\mathcal{D}$.

This kind of blow-up behavior must occur if a solution cannot be continued to the whole real line.

Theorem on Solution Blow-Up

Suppose $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on a closed set containing the open set $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$. Let $(t_0, x_0) \in \mathcal{D}$, and let (T_-, T_+) be the maximal interval of existence of the solution of the IVP

$$x' = f(t, x), \quad x(t_0) = x_0 \ .$$

If $T_+ < +\infty$ $(T_- > -\infty)$, then for any compact set $K \subset \mathcal{D}$, there exists a $T < T_+$ $(T_- < T)$ for which $(t, x(t)) \notin K$ for $t > T$ $(t < T)$.

Proof of Theorem on Solution Blow-Up

If not, $\exists\, t_j \to T_+$ with $(t_j, x(t_j))) \in K$ for all j. By taking a subsequence, we may assume that $x(t_j)$ also converges to $x_+ \in \mathbb{F}^n$, and

$$(t_j, x(t_j)) \to (T_+, x_+) \in K \subset \mathcal{D}.$$

We can thus choose $r > 0$, $\tau > 0$, $N \in \mathbb{N}$ such that

$$\mathcal{S} = \bigcup_{j=N}^{\infty} \{(t, x) : |t - t_j| \leq \tau, |x - x(t_j)| \leq r\} \subset \mathcal{D}.$$

Since $\mathcal{S} \subset \mathcal{D}$ is bounded, there is an M for which $|f(t, x)| \leq M$ on $\mathcal{S}$. By the local existence theorem, the solution of $x' = f(t, x)$ starting at the initial point $(t_j, x(t_j))$ exists for a time interval of length

$$T' \equiv \min\left\{\tau, \frac{r}{M}\right\},$$

independent of i. Choose j for which $t_j > t_+ - T'$. Then $(t, x(t))$ exists in $\mathcal{D}$ beyond time T_+, which is a contradiction.

Autonomous Systems

The ODE $x'(t) = f(t, x)$ is called an **autonomous system** if $f(t, x)$ is independent of t, i.e., the ODE is of the form

$$x' = f(x).$$

Remarks.

(1) Time translates of solutions of an autonomous system are again solutions:
$x(t)$ a solution $\implies$ $x(t - c)$ is a solution for any constant c.

(2) Any ODE $x' = f(t, x)$ is equivalent to an autonomous system. Define "$x_{n+1} = t$" and set

$$\widetilde{x} = (x_{n+1}, x) \in \mathbb{F}^{n+1}$$

$$\widetilde{x}' = \widetilde{f}(\widetilde{x}) = \widetilde{f}(x_{n+1}, x) = \begin{bmatrix} 1 \\ f(x_{n+1}, x) \end{bmatrix} \in \mathbb{F}^{n+1}$$

and consider the autonomous IVP

$$\widetilde{x}' = \widetilde{f}(\widetilde{x}), \quad \widetilde{x}(t_0) = \begin{bmatrix} t_0 \\ x_0 \end{bmatrix}.$$

This IVP is equivalent to the IVP

$$x' = f(t, x), \quad x(t_0) = x_0.$$

Continuation for Autonomous Systems

Suppose $f(x)$ is defined and locally Lipschitz continuous on an open set $\mathcal{U} \subset \mathbb{F}^n$. Take $\mathcal{D} = \mathbb{R} \times \mathcal{U}$. Suppose $T_+ < \infty$ and C is a compact subset of $\mathcal{U}$. Take $K = [t_0, T_+] \times C$ in the ODE Blow-Up Theorem. Then

$$\exists\, T < T_+ \quad \text{such that} \quad x(t) \notin C \quad \text{for} \quad T < t < T_+.$$

In this case we say that

$$x(t) \to \partial\mathcal{U} \cup \{\infty\} \quad \text{as} \quad t \to (T_+)^-,$$

meaning that

$$(\forall\, C^{\text{compact}} \subset \mathcal{U})(\exists\, T < T_+) \quad \text{such that for} \quad t \in (T, T_+),\ x(t) \notin C.$$

Stated briefly, eventually $x(t)$ stays out of any given compact set.

Continuation of Linear Systems

Consider the linear IVP

$$x'(t) = A(t)x(t) + b(t), \quad x(t_0) = x_0 \quad \text{on} \quad (a,\ b) \quad \textit{with} \quad t_0 \in (a,\ b),$$

where $A(t) \in \mathbb{F}^{n\times n}$ and $b(t) \in \mathbb{F}^n$ are continuous on (a, b). Let $\mathcal{D} = (a, b) \times \mathbb{F}^n$. Then

$$f(t, x) = A(t)x + b(t) \in (C, \mathrm{Lip}_{\mathrm{loc}}) \quad \text{on} \quad \mathcal{D}.$$

Moreover, for c, d satisfying

$$a < c \le t_0 \le d < b,$$

f is uniformly Lipschitz continuous with respect to x on $[c, d] \times \mathbb{F}^n$,

$$\text{take} \quad L = \max_{c\le t\le d} |A(t)|.$$

The Picard global existence theorem implies there is a solution of the IVP on $[c, d]$, which is unique by the uniqueness theorem for locally Lipschitz f. This implies that $T_- = a$ and $T_+ = b$.

Dependence of Solutions on Initial Values

Definition.
We say that $x(t)$ is an ϵ-**approximate solution** of the DE

$$x' = f(t, x) \quad \text{on} \quad I \subset \mathbb{R}$$

if

$$|x'(t) - f(t, x(t))| \leq \epsilon \qquad (\forall\, t \in I).$$

Fundamental Estimate

Let $f(t,x)$ be continuous in t and x, and uniformly Lipschitz continuous in x with Lipschitz constant L. Consider the DE

$$(*) \qquad\qquad x' = f(t,x)\ .$$

Let $\epsilon_1, \epsilon_2 > 0$, and suppose

$$x_i(t) \quad \text{is an } \epsilon_i\text{-approximate solution of (*) on } I,\ i = 1, 2..$$

Given $t_0 \in I$, suppose that

$$|x_1(t_0) - x_2(t_0)| \leq \delta\ .$$

Then, for $t \in I$,

$$|x_1(t) - x_2(t)| \leq \delta e^{L|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{L}(e^{L|t-t_0|} - 1)\ .$$

Remarks on the Fundamental Estimate

$$|x_1(t) - x_2(t)| \leq \delta e^{L|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{L}(e^{L|t-t_0|} - 1) \ .$$

Remarks.

(1) The first term on the RHS bounds the difference between the solutions of the IVPs with initial values $x_1(t_0)$ and $x_2(t_0)$ at t_0.

(2) The second term on the RHS accounts for the fact that $x_1(t)$ and $x_2(t)$ are only approx. solutions: note that this term is 0 at $t = t_0$.

(3) If $\epsilon_1 = \epsilon_2 = \delta = 0$, we can again recover the uniqueness theorem for Lipschitz f.

Proof of the Fundamental Estimate

We may assume $\epsilon_1, \epsilon_2, \delta > 0$ (otherwise, take limits as $\epsilon_1 \to 0^+$, $\epsilon_2 \to 0^+$, $\delta \to 0^+$). Also for simplicity, we may assume $t_0 = 0$ and we are considering $t \geq 0$ (do time reversal for $t \leq 0$). Let

$$u(t) = |x_1(t) - x_2(t)|^2 = \langle x_1 - x_2, x_2 - x_2 \rangle.$$

Then

$$\begin{aligned} u' &= 2\mathcal{R}e\langle x_1 - x_2, x_1' - x_2' \rangle \leq 2|x_1 - x_2| \cdot |x_1' - x_2'| \\ &= 2|x_1 - x_2|\,|x_1' - f(t, x_1) - (x_2' - f(t, x_2)) + f(t, x_1) - f(t, x_2)| \\ &\leq 2|x_1 - x_2|(\epsilon_1 + \epsilon_2 + L|x_1 - x_2|) \\ \\ &= 2Lu + 2\epsilon\sqrt{u}, \end{aligned}$$

where $\epsilon = \epsilon_1 + \epsilon_2$.

Proof of the Fundamental Estimate

We want to use the Comparison Theorem to compare u to the solution v of

$$v' = 2Lv + 2\epsilon\sqrt{v}, \;\; v(0) = \delta^2 > 0.$$

But $\widetilde{f}(v) \equiv 2Lv + 2\epsilon\sqrt{v}$ is not Lipschitz on $v \in [0, \infty)$; it is, however, for a fixed $\delta > 0$, uniformly Lipschitz on $v \in [\delta^2, \infty)$ since $\frac{d\widetilde{f}}{dv} = 2L + \frac{\epsilon}{\sqrt{v}}$ is bounded for $v \in [\delta^2, \infty)$, and C^1 functions with bounded derivatives are uniformly Lipschitz

$$|\widetilde{f}(v_1) - \widetilde{f}(v_2)| = \left| \int_{v_2}^{v_1} \frac{d\widetilde{f}}{dv} dv \right| \leq (\sup \left| \frac{d\widetilde{f}}{dv} \right|)|v_1 - v_2|).$$

Proof of the Fundamental Estimate

Although $u(t)$ may leave $[\delta^2, \infty)$, in the proof of the Comparison Theorem we only need $\widetilde{f}$ to be Lipschitz to conclude that $u > v$ cannot occur.

Note that since $v' \geq 0$, $v(t)$ stays in $[\delta^2, \infty)$ for $t \geq 0$. So the Comparison Theorem does apply, and we conclude that $u \leq v$ for $t \geq 0$.

To solve for v, let $v = w^2$. Then

$$2ww' = (w^2)' = v' = 2Lw^2 + 2\epsilon w.$$

Since $w > 0$, we get $w' = Lw + \epsilon$, $w(0) = \delta$, whose solution is

$$w = \delta e^{Lt} + \frac{\epsilon}{L}(e^{Lt} - 1).$$

Since $|x_1 - x_2| = \sqrt{u} \leq \sqrt{v} = w$, the estimate follows.

Corollary: Uniform Convergence of Solutions to IVPs

Corollary: For $j \geq 1$, suppose $x_j(t)$ is a solution to the DE

$$x_j' = f_j(t, x_j) \quad \text{on} \quad I := [a, b],$$

and suppose $x(t)$ is a solution to the DE

$$x' = f(t, x) \quad \text{on} \quad I,$$

where the f_j's and f are continuous in t and x and f is Lipschitz in x.

If $f_j \to f$ uniformly on $[a, b] \times \mathbb{F}^n$ and there is a $t_0 \in I$ such that $x_j(t_0) \to x(t_0)$, then

$$x_j(t) \to x(t) \quad \text{uniformly on} \quad [a, b].$$

Proof: By the Fundamental Estimate, with $x_1 = x_j$ and $x_2 = x$,

$$|x_j(t) - x(t)| \leq |x_j(t_0) - x(t_0)| e^{L|t-t_0|} + \frac{\epsilon_j}{L}(e^{L|t-t_0|} - 1),$$

where $\epsilon_j := \sup_{t \in [a,b]} |f_j(t, x_j(t)) - f(t, x_j(t))|$.