Linear Analysis
Lecture 22

The Cauchy-Peano Existence Theorem

Let $I = [t_0, t_0 + \beta]$ and

$$\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq r\},$$

and suppose $f(t, x)$ is continuous on $I \times \Omega$.
Then there exists a solution $x_*(t)$ of the integral equation

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$$

in $C(I_\alpha)$ where $I_\alpha = [t_0, t_0 + \alpha]$,

$$\alpha = \min\left(\beta, \frac{r}{M}\right),$$

and

$$M = \max_{(t,x) \in I \times \Omega} |f(t, x)|,$$

and so $x_*(t)$ is a C^1 solution of the initial value problem

$$IVP: \qquad x' = f(t, x), \quad x(t_0) = x_0$$

in I_α.

Uniqueness

Uniqueness theorems are typically proved by comparison theorems for solutions of scalar differential equations, or by inequalities.
The most fundamental of these inequalities is **Gronwall's inequality**.
Recall that a first-order linear scalar IVP

$$u' = a(t)u + b(t), \quad u(t_0) = u_0.$$

Rewrite this as

$$u' - a(t)u = b(t), \quad u(t_0) = u_0,$$

and multiply by the integrating factor

$$e^{-\int_{t_0}^{t} a(\ell)\,d\ell},$$

to get

$$u' e^{-\int_{t_0}^{t} a(\ell)\,d\ell} - a(t) e^{-\int_{t_0}^{t} a(\ell)\,d\ell} u = e^{-\int_{t_0}^{t} a(\ell)\,d\ell} b(t), \quad u(t_0) = u_0.$$

That is,

$$\frac{d}{dt}\left(e^{-\int_{t_0}^{t} a(\ell)\,d\ell} u(t) \right) = e^{-\int_{t_0}^{t} a(\ell)\,d\ell} b(t).$$

Uniqueness

Now integrate from t_0 to t:

$$\begin{aligned} e^{-\int_{t_0}^{t} a(\ell)\,d\ell} u(t) - u_0 &= \int_{t_0}^{t} \frac{d}{ds}\left(e^{-\int_{t_0}^{s} a(\ell)\,d\ell} u(s) \right) ds \\ &= \int_{t_0}^{t} e^{-\int_{t_0}^{s} a(\ell)\,d\ell} b(s)\,ds \\ &\Rightarrow \\ u(t) &= u_0\, e^{\int_{t_0}^{t} a(\ell)\,d\ell} + \int_{t_0}^{t} e^{\int_{s}^{t} a(\ell)\,d\ell} b(s)\,ds\ . \end{aligned}$$

Since $f(t) \leq g(t)$ on $[c, d]$ implies

$$\int_c^d f(t)\,dt \leq \int_c^d g(t)\,dt,$$

the identical argument with "$=$" replaced by "$\leq$" gives Gronwall's inequality.

Theorem (Gronwall's Inequality - Differential Form)

Let $I = [t_0, t_1]$. Suppose $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ are continuous, and suppose $u : I \to \mathbb{R}$ is in $C^1(I)$ and satisfies

$$u'(t) \leq a(t)u(t) + b(t) \quad for \quad t \in I,$$

and $u(t_0) = u_0$. Then

$$u(t) \leq u_0 e^{\int_{t_0}^{t} a(\ell)\,d\ell} + \int_{t_0}^{t} e^{\int_{s}^{t} a(\ell)\,d\ell} b(s)\,ds.$$

Remarks

(1) Thus a solution of the differential inequality is bounded above by the solution of the differential equality.

(2) Clearly, Gronwall's Inequality still holds if u is only continuous and piecewise C^1, and $a(t)$ and $b(t)$ are only piecewise continuous.

Theorem: (Uniqueness for Locally Lipschitz f)

Suppose for $\alpha > 0,\ r > 0$, $f(t,x)$ is in (C, Lip) on $I_\alpha \times \overline{B_r(x_0)}$.

Further suppose both $x(t)$ and $y(t)$ map I_α into $\overline{B_r(x_0)}$ and are C^1 solutions of the IVP

$$x' = f(t,x)\,; \quad x(t_0) = x_0 \quad \text{on} \quad I_\alpha,$$

where $I_\alpha = [t_0, t_0 + \alpha]$.

Then $x(t) = y(t)$ for $t \in I_\alpha$.

Proof of Uniqueness for Locally Lipschitz f

Set $\quad u(t) = |x(t) - y(t)|^2 = \langle x(t) - y(t), x(t) - y(t) \rangle$
(in the Euclidean inner product on $\mathbb{F}^n$). Then $u : I_\alpha \to [0, \infty)$ and $u \in C^1(I_\alpha)$ and for $t \in I_\alpha$,

$$\begin{aligned} u' &= \langle x - y, x' - y' \rangle + \langle x' - y', x - y \rangle \\ &= 2\mathcal{R}e\langle x - y, x' - y' \rangle \\ &\leq 2|\langle x - y, x' - y' \rangle| \\ &= 2|\langle x - y, (f(t,x) - f(t,y)) \rangle| \\ &\leq 2|x - y| \cdot |f(t,x) - f(t,y)| \\ &\leq 2L|x - y|^2 = 2Lu \ . \end{aligned}$$

Thus $u' \leq 2Lu$ on I_α and

$$u(t_0) = x(t_0) - y(t_0) = x_0 - x_0 = 0.$$

By Gronwall's inequality,

$$u(t) \leq u_0 e^{2Lt} = 0 \quad \text{on} \quad I_\alpha,$$

since $u(t) \geq 0$ on I_α, we have

$$u(t) \equiv 0 \quad \text{on} \quad I_\alpha. \qquad \square$$

Corollary.

(i) The same result holds if $I_\alpha = [t_0 - \alpha, t_0]$.

(ii) The same result holds if $I_\alpha = [t_0 - \alpha, t_0 + \alpha]$.

Proof: For (i), let

$$\widetilde{x}(t) = x(2t_0 - t),\ \widetilde{y}(t) = y(2t_0 - t),\ \text{and}$$

$$\widetilde{f}(t, x) = -f(2t_0 - t, x).$$

Then $\widetilde{f}$ is in (C, Lip) on $[t_0, t_0 + \alpha] \times \overline{B_r(x_0)}$, and $\widetilde{x}$ and $\widetilde{y}$ both satisfy

$$x' = \widetilde{f}(t, x); \quad x'(t_0) = x_0 \quad \text{on} \quad [t_0, t_0 + \alpha].$$

So by the Theorem, $\widetilde{x}(t) = \widetilde{y}(t)$ for $t \in [t_0, t_0 + \alpha]$, i.e., $x(t) = y(t)$ for $t \in [t_0 - \alpha, t_0]$. Now (ii) follows immediately by applying the Theorem in $[t_0, t_0 + \alpha]$ and applying (ii) in $[t_0 - \alpha, t_0]$. □

Remark. The idea used in the proof of (i) is often called "time-reversal." The important part is that $\widetilde{x}(t) = x(c - t)$, for some constant c, so that $\widetilde{x}'(t) = -x'(c - t)$. The choice of $c = 2t_0$ is convenient but not essential.

Local Lipschitz Contnuity

Before stating our main uniqueness result, we introduce a local form of Lipschitz continuity of the function $f(t, x)$ in the x argument.

Definition. Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$. We say that $f(t, x)$ mapping $\mathcal{D}$ into $\mathbb{F}^n$ is **locally Lipschitz continuous with respect to x** if

$$\forall\, (t_1, x_1) \in \mathcal{D}, \quad \exists \quad \alpha > 0, \quad r > 0 \quad and \quad L > 0$$

for which $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D}$ and

$$(\forall\, t \in [t_1 - \alpha, t_1 + \alpha])\ (\forall\, x, y \in \overline{B_r(x_1)})$$

$$|f(t, x) - f(t, y)| \leq L|x - y|\ ,$$

i.e., f is uniformly Lipschitz continuous with respect to x in

$$[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)}.$$

We say $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ (not a standard notation) on $\mathcal{D}$ if f is continuous on $\mathcal{D}$ and locally Lipschitz continuous wrt x on $\mathcal{D}$.

Local Lipschitz Contnuity: Example

Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$. Suppose $f(t,x)$ maps $\mathcal{D}$ into $\mathbb{F}^n$, f is continuous on $\mathcal{D}$, and

$$\text{for} \quad 1 \leq i,\ j \leq n, \quad \frac{\partial f_i}{\partial x_j} \quad \text{exists and is continuous in} \quad \mathcal{D},$$

i.e., f is continuous on $\mathcal{D}$ and C^1 with respect to x on $\mathcal{D}$. Then $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$.

Main Uniqueness Theorem

Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$, and suppose

(a) $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$,

(b) $(t_0, x_0) \in \mathcal{D}$,

(c) $I \subset \mathbb{R}$ is an interval containing t_0 (which may be open or closed at either end), and

(d) $x(t)$ and $y(t)$ are both solutions of the IVP

$$x' = f(t, x); \quad x(t_0) = x_0 \quad \text{in} \quad C^1(I)$$

which satisfy

$$(t, x(t)) \in \mathcal{D} \quad \text{and} \quad (t, y(t)) \in \mathcal{D} \qquad \forall\ t \in I.$$

Then $x(t) \equiv y(t)$ on I.

We first show $x(t) \equiv y(t)$ on $\{t \in I : t \geq t_0\}$. If not, let

$$t_1 = \inf\{t \in I : t \geq t_0 \text{ and } x(t) \neq y(t)\}.$$

Then $x(t) = y(t)$ on $[t_0, t_1)$ so by continuity $x(t_1) = y(t_1)$ (if $t_1 = t_0$, this is obvious). By continuity and the openness of $\mathcal{D}$ (as $(t_1, x(t_1)) \in \mathcal{D}$), $\exists\ \alpha > 0$ and $r > 0$ such that $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D}$, f is uniformly Lipschitz continuous with respect to x in

$$[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)},$$

and

$$x(t) \in \overline{B_r(x_1)} \quad \text{and} \quad y(t) \in \overline{B_r(x_1)} \quad \forall\ t \in I \cap [t_1 - \alpha, t_1 + \alpha].$$

By the previous theorem, $x(t) \equiv y(t)$ in $I \cap [t_1 - \alpha, t_1 + \alpha]$, contradicting the definition of t_1. Hence

$$x(t) \equiv y(t) \quad \text{on} \quad \{t \in I : t \geq t_0\}.$$

Similarly,

$$x(t) \equiv y(t) \quad \text{on} \quad \{t \in I : t \leq t_0\}.$$

Hence $x(t) \equiv y(t)$ on I. □

Remark. t_0 is allowed to be the left or right endpoint of I.

Theorem. Let $n = 1$, $\mathbb{F} = \mathbb{R}$, and suppose $f(t, u)$ is continuous in t and Lipschitz continuous in u.
Assume $u(t)$, $v(t)$ are C^1 for $t \geq t_0$ (on an interval $[t_0, b)$ or $[t_0, b]$) and satisfy

$$u'(t) \leq f(t, u(t)), \qquad v'(t) = f(t, v(t))$$

and $u(t_0) \leq v(t_0)$. Then

$$u(t) \leq v(t) \quad \text{for} \quad t \geq t_0.$$

Comparison Theorem for Real Scalar Equations: Proof

If to the contrary $u(T) > v(T)$ for some $T > t_0$, then set

$$t_1 = \sup\{t : t_0 \leq t < T \quad \text{and} \quad u(t) \leq v(t)\}.$$

Then

$$t_0 \leq t_1 < T, \quad u(t_1) = v(t_1), \quad \text{and} \quad u(t) > v(t) \quad \text{for} \quad t_1 < t \leq T$$

(by continuity of $u - v$). For

$$t_1 \leq t \leq T, \quad |u(t) - v(t)| = u(t) - v(t),$$

so we have

$$(u - v)' \leq f(t, u) - f(t, v) \leq L|u - v| = L(u - v).$$

By Gronwall's inequality applied to $u - v$ on $[t_1, T]$, with

$$(u - v)(t_1) = 0, \; a(t) \equiv L, \; b(t) \equiv 0,$$

$(u - v)(t) \leq 0$ on $[t_1, T]$, a contradiction.

Remarks.

(1) As with the differential form of Gronwall's inequality a solution of the differential inequality $u' \leq f(t, u)$ is bounded above by the solution of the equality (i.e., the DE $v' = f(t, v)$).

(2) It can be shown under the same hypotheses that if $u(t_0) < v(t_0)$, then $u(t) < v(t)$ for $t \geq t_0$.

(3) Caution: It may happen that $u'(t) > v'(t)$ for some $t \geq t_0$: $u(t) \leq v(t) \not\Rightarrow u'(t) \leq v'(t)$.

Comparison Theorem for Real Scalar Equations: Corollary

Corollary. Let $n = 1$, $\mathbb{F} = \mathbb{R}$. Suppose $f(t,u) \le g(t,u)$ are continuous in t and u, and one of them is Lipschitz continuous in u. Suppose also that $u(t)$, $v(t)$ are C^1 for $t \ge t_0$ (on $[t_0, b)$ or $[t_0, b]$) and satisfy

$$u' = f(t,u), \;\; v' = g(t,v), \quad \text{and} \quad u(t_0) \le v(t_0).$$

Then

$$u(t) \le v(t) \quad \text{for} \quad t \ge t_0.$$

Proof: Suppose first that g satisfies the Lipschitz condition. Then

$$u' = f(t,u) \le g(t,u).$$

Now apply the theorem. If f satisfies the Lipschitz condition, apply the first part of this proof to

$$\widetilde{u}(t) \equiv -v(t), \;\; \widetilde{v}(t) \equiv -u(t), \;\; \widetilde{f}(t,u) = -g(t,-u), \;\; \widetilde{g}(t,u) = -f(t,-u).$$

□

Remark. Again, if $u(t_0) < v(t_0)$, then $u(t) < v(t)$ for $t \ge t_0$.

We consider two kinds of results
(1) *local continuation* (no Lipschitz condition on f)
(2) *global continuation* (for locally Lipschitz f)

Local Continuation (Continuation at a Point)

Assume $x(t)$ is a solution of the DE $x' = f(t,x)$ on an interval I and f is continuous on a subset $\mathcal{S} \subset \mathbb{R} \times \mathbb{F}^n$ containing $\{(t, x(t)) : t \in I\}$.

Case 1: I is closed at the right end, i.e., $I = (-\infty, b]$, $[a, b]$, or $(a, b]$. Assume further that $(b, x(b))$ is in the interior of $\mathcal{S}$. Then the solution can be extended (by Cauchy-Peano) to an interval with right end $b + \beta$ for some $\beta > 0$. This is done by solving the IVP

$$x' = f(t,x) \quad \text{with initial value } x(b) \text{ at } t = b$$

on an interval $[b, b+\beta]$. To show that the continuation is C^1 at $t = b$, note that the extended $x(t)$ satisfies the integral equation

$$x(t) = x(b) + \int_b^t f(s, x(s))\,ds \quad \text{on } I \bigcup [b, b+\beta].$$

Note we do not assume Lipschitz continuity.

Continuation of Solutions in Time; Case 2

Case 2: I is open at the right end, i.e., $I = (-\infty, b)$, $[a, b)$, or (a, b) with $b < \infty$. Assume further that $f(t, x(t))$ is **bounded** on $[t_0, b)$ for some $t_0 < b$ with $[t_0, b) \subset I$, say $|f(t, x(t))| \leq M$ on $[t_0, b)$.
In this case the integral equation

$$(*) \qquad x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\,ds$$

holds for $t \in I$. In particular, for $t_0 \leq \tau \leq t < b$,

$$|x(t) - x(\tau)| = \left| \int_{\tau}^{t} f(s, x(s))\,ds \right| \leq \int_{\tau}^{t} |f(s, x(s))|\,ds \leq M|t - \tau|.$$

Thus, for any sequence $t_n \uparrow b$, $\{x(t_n)\}$ is Cauchy. This implies $\lim_{t \to b^-} x(t)$ exists; call it $x(b^-)$. So $x(t)$ has a **continuous** extension from I to $I \cup \{b\}$.

Comments

- If in addition $(b, x(b^-))$ is in $\mathcal{S}$, then $(*)$ holds on $I \cup \{b\}$ as well, so $x(t)$ is a C^1 **solution** of $x' = f(t, x)$ on $I \cup \{b\}$.

- If $(b, x(b^-))$ is in the interior of $\mathcal{S}$, we are back in Case 1 and can extend the solution $x(t)$ beyond $t = b$.

- The assumption that $f(t, x(t))$ is bounded on $[t_0, b)$ can be restated with a slightly different emphasis: for some $t_0 \in I$, $\{(t, x(t)) : t_0 \le t < b\}$ stays within a subset of $\mathcal{S}$ on which f is bounded. For example, if $\{(t, x(t)) : t_0 \le t < b\}$ stays within a compact subset of $\mathcal{S}$, this condition is satisfied.

- The technique of Case 1 can be applied to I is closed at the left end.

- The technique of Case 2 can be applied to I is open at the left end.

Global Continuation

Assume $f(t,x)$ is continuous on an open set $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$ and is locally Lipschitz continuous with respect to x on $\mathcal{D}$. Write $f \in (C, \text{Lip}_{\text{loc}})$ on $\mathcal{D}$.

Let $(t_0, x_0) \in \mathcal{D}$ and consider the IVP

$$x' = f(t,x), \quad x(t_0) = x_0.$$

It has been shown that a unique solutions exist on both $[t_0, t_0 + \alpha_+)$ and $(-\alpha_- + t_0, t_0]$, and that this gives a unique solution on $(-\alpha_- + t_0,\ t_0 + \alpha_+)$ for some $\alpha_+, \alpha_- > 0$. Set

$$\begin{aligned} T_+ &= \sup\{t > t_0 : \exists \text{ a solution of IVP on } [t_0, t)\}, \quad \text{and} \\ T_- &= \inf\{t < t_0 : \exists \text{ a solution of IVP on } (t,\ t_0]\}. \end{aligned}$$

(T_-, T_+) is the maximal interval of existence of the solution of the IVP. It is possible that $T_+ = \infty$ and/or $T_- = -\infty$.

The maximal interval (T_-, T_+) must be open: if the solution could be extended to T_+ (or T_-), this would contradict the local continuation results since $\mathcal{D}$ is open. Ideally, $T_+ = +\infty$ and $T_- = -\infty$.

Global Continuation

Another posibility is if $f(t, x)$ is not defined for $t \geq T_+$. For example, if $a(t) = \frac{1}{1-t}$, and $x'(t) = a(t)$. Here we don't expect the solution to exist beyond $t = 1$.

But less desirable behavior can occur.

For example, for the IVP:

$$x' = x^2,\ x(0) = x_0 > 0,\ t_0 = 0,$$

and $\mathcal{D} = \mathbb{R} \times \mathbb{R}$. The solution $x(t) = (x_0^{-1} - t)^{-1}$ blows up at $T_+ = 1/x_0$ (note that $T_- = -\infty$). Observe that $x(t) \to \infty$ as $t \to (T_+)^-$. So the solution does not just "stop" in the interior of $\mathcal{D}$.

This kind of blow-up behavior must occur if a solution cannot be continued to the whole real line.

Theorem on Solution Blow-Up

Suppose $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on an open set $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$. Let $(t_0, x_0) \in \mathcal{D}$, and let (T_-, T_+) be the maximal interval of existence of the solution of the IVP

$$x' = f(t, x), \quad x(t_0) = x_0 \ .$$

If $T_+ < +\infty$ ($T_- > -\infty$), then for any compact set $K \subset \mathcal{D}$, there exists a $T < T_+$ ($T_- < T$) for which $(t, x(t)) \notin K$ for $t > T$ ($t < T$).

Proof of Theorem on Solution Blow-Up

If not, $\exists\, t_j \to T_+$ with $(t_j, x(t_j))) \in K$ for all j. By taking a subsequence, we may assume that $x(t_j)$ also converges to $x_+ \in \mathbb{F}^n$, and

$$(t_j, x(t_j)) \to (T_+, x_+) \in K \subset \mathcal{D}.$$

We can thus choose $r > 0$, $\tau > 0$, $N \in \mathbb{N}$ such that

$$\mathcal{S} = \bigcup_{j=N}^{\infty} \{(t,x) : |t - t_j| \leq \tau, |x - x(t_j)| \leq r\} \subset \mathcal{D}.$$

Since $\mathcal{D}$ is compact, there is an M for which $|f(t,x)| \leq M$ on $\mathcal{S}$. By the local existence theorem, the solution of $x' = f(t,x)$ starting at the initial point $(t_j, x(t_j))$ exists for a time interval of length

$$T' \equiv \min \left\{ \tau, \frac{r}{M} \right\},$$

independent of i. Choose j for which $t_j > t_+ - T'$. Then $(t, x(t))$ exists in $\mathcal{D}$ beyond time T_+, which is a contradiction.

Autonomous Systems

The ODE $x'(t) = f(t, x)$ is called an **autonomous system** if $f(t, x)$ is independent of t, i.e., the ODE is of the form

$$x' = f(x).$$

Remarks.

(1) Time translates of solutions of an autonomous system are again solutions:
$x(t)$ a solution $\implies$ $x(t-c)$ is a solution for any constant c.

(2) Any ODE $x' = f(t, x)$ is equivalent to an autonomous system. Define "$x_{n+1} = t$" and set

$$\widetilde{x} = (x_{n+1}, x) \in \mathbb{F}^{n+1}$$

$$\widetilde{x}' = \widetilde{f}(\widetilde{x}) = \widetilde{f}(x_{n+1}, x) = \begin{bmatrix} 1 \\ f(x_{n+1}, x) \end{bmatrix} \in \mathbb{F}^{n+1}$$

and consider the autonomous IVP

$$\widetilde{x}' = \widetilde{f}(\widetilde{x}), \quad \widetilde{x}(t_0) = \begin{bmatrix} t_0 \\ x_0 \end{bmatrix}.$$

This IVP is equivalent to the IVP

$$x' = f(t, x), \quad x(t_0) = x_0.$$

Continuation for Autonomous Systems

Suppose $f(x)$ is defined and locally Lipschitz continuous on an open set $\mathcal{U} \subset \mathbb{F}^n$. Take $\mathcal{D} = \mathbb{R} \times \mathcal{U}$. Suppose $T_+ < \infty$ and C is a compact subset of $\mathcal{U}$. Take $K = [t_0, T_+] \times C$ in the ODE Blow-Up Theorem. Then

$$\exists\, T < T_+ \quad \text{such that} \quad x(t) \notin C \quad \text{for} \quad T < t < T_+.$$

In this case we say that

$$x(t) \to \partial\mathcal{U} \cup \{\infty\} \quad \text{as} \quad t \to (T_+)^-,$$

meaning that

$$(\forall\, C^{\text{compact}} \subset \mathcal{U})(\exists\, T < T_+) \quad \text{such that for} \quad t \in (T, T_+),\ x(t) \notin C.$$

Stated briefly, eventually $x(t)$ stays out of any given compact set.

Continuation of Linear Systems

Consider the linear IVP

$$x'(t) = A(t)x(t) + b(t), \quad x(t_0) = x_0 \quad \text{on} \quad (a,\, b) \quad with \quad t_0 \in (a,\, b),$$

where $A(t) \in \mathbb{F}^{n \times n}$ and $b(t) \in \mathbb{F}^n$ are continuous on (a, b). Let $\mathcal{D} = (a, b) \times \mathbb{F}^n$. Then

$$f(t, x) = A(t)x + b(t) \in (C, \mathrm{Lip}_{\mathrm{loc}}) \quad \text{on} \quad \mathcal{D}.$$

Moreover, for c, d satisfying

$$a < c \leq t_0 \leq d < b,$$

f is uniformly Lipschitz continuous with respect to x on $[c, d] \times \mathbb{F}^n$,

$$\text{take} \quad L = \max_{c \leq t \leq d} |A(t)|.$$

The Picard global existence theorem implies there is a solution of the IVP on $[c, d]$, which is unique by the uniqueness theorem for locally Lipschitz f. This implies that $T_- = a$ and $T_+ = b$.

Definition.
We say that $x(t)$ is an ϵ-**approximate solution** of the DE

$$x' = f(t,x) \quad \text{on} \quad I \subset \mathbb{R}$$

if

$$|x'(t) - f(t, x(t))| \leq \epsilon \qquad (\forall\, t \in I).$$

Fundamental Estimate

Let $f(t,x)$ be continuous in t and x, and uniformly Lipschitz continuous in x with Lipschitz constant L. Consider the DE

$$(*) \qquad\qquad x' = f(t,x) \ .$$

Let $\epsilon_1, \epsilon_2 > 0$, and suppose

$x_i(t)$ is an ϵ_i-approximate solution of (*) on I, $i = 1, 2.$.

Given $t_0 \in I$, suppose that

$$|x_1(t_0) - x_2(t_0)| \leq \delta \ .$$

Then, for $t \in I$,

$$|x_1(t) - x_2(t)| \leq \delta e^{L|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{L}(e^{L|t-t_0|} - 1) \ .$$

Remarks on the Fundamental Estimate

$$|x_1(t) - x_2(t)| \leq \delta e^{L|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{L}(e^{L|t-t_0|} - 1) \ .$$

Remarks.

(1) The first term on the RHS bounds the difference between the solutions of the IVPs with initial values $x_1(t_0)$ and $x_2(t_0)$ at t_0.

(2) The second term on the RHS accounts for the fact that $x_1(t)$ and $x_2(t)$ are only approx. solutions: note that this term is 0 at $t = t_0$.

(3) If $\epsilon_1 = \epsilon_2 = \delta = 0$, we can again recover the uniqueness theorem for Lipschitz f.

Proof of the Fundamental Estimate

We may assume $\epsilon_1, \epsilon_2, \delta > 0$ (otherwise, take limits as $\epsilon_1 \to 0^+$, $\epsilon_2 \to 0^+$, $\delta \to 0^+$). Also for simplicity, we may assume $t_0 = 0$ and we are considering $t \geq 0$ (do time reversal for $t \leq 0$). Let

$$u(t) = |x_1(t) - x_2(t)|^2 = \langle x_1 - x_2, x_2 - x_2 \rangle.$$

Then

$$\begin{aligned}
u' &= 2\mathcal{R}e\langle x_1 - x_2, x_1' - x_2' \rangle \leq 2|x_1 - x_2| \cdot |x_1' - x_2'| \\
&= 2|x_1 - x_2|\,|x_1' - f(t, x_1) - (x_2' - f(t, x_2)) + f(t, x_1) - f(t, x_2)| \\
&\leq 2|x_1 - x_2|(\epsilon_1 + \epsilon_2 + L|x_1 - x_2|) \\
\\
&= 2Lu + 2\epsilon\sqrt{u},
\end{aligned}$$

where $\epsilon = \epsilon_1 + \epsilon_2$.

Proof of the Fundamental Estimate

We want to use the Comparison Theorem to compare u to the solution v of

$$v' = 2Lv + 2\epsilon\sqrt{v}, \ \ v(0) = \delta^2 > 0.$$

But $\widetilde{f}(v) \equiv 2Lv + 2\epsilon\sqrt{v}$ is not Lipschitz on $v \in [0, \infty)$; it is, however, for a fixed $\delta > 0$, uniformly Lipschitz on $v \in [\delta^2, \infty)$ since $\frac{d\widetilde{f}}{dv} = 2L + \frac{\epsilon}{\sqrt{v}}$ is bounded for $v \in [\delta^2, \infty)$, and C^1 functions with bounded derivatives are uniformly Lipschitz

$$(|\widetilde{f}(v_1) - \widetilde{f}(v_2)| = \left| \int_{v_2}^{v_1} \frac{d\widetilde{f}}{dv} dv \right| \leq (\sup |\widetilde{f}(v)|)|v_1 - v_2|).$$

Proof of the Fundamental Estimate

Although $u(t)$ may leave $[\delta^2, \infty)$, in the proof of the Comparison Theorem we only need $\widetilde{f}$ to be Lipschitz to conclude that $u > v$ cannot occur.

Note that since $v' \geq 0$, $v(t)$ stays in $[\delta^2, \infty)$ for $t \geq 0$. So the Comparison Theorem does apply, and we conclude that $u \leq v$ for $t \geq 0$.

To solve for v, let $v = w^2$. Then

$$2ww' = (w^2)' = v' = 2Lw^2 + 2\epsilon w.$$

Since $w > 0$, we get $w' = Lw + \epsilon$, $w(0) = \delta$, whose solution is

$$w = \delta e^{Lt} + \frac{\epsilon}{L}(e^{Lt} - 1).$$

Since $|x_1 = x_2| = \sqrt{u} \leq \sqrt{v} = w$, the estimate follows.