Linear Analysis
Lecture 21

The Picard Iteration

We now study the fixed point iteration based on the function

$$(g(x))(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \ .$$

used in the previous existence and uniqueness theorem for the integral equation (IE). If we choose the initial iterate to be $x_0(t) \equiv x_0$, we obtain the classical Picard Iteration:

$$\begin{cases} x_0(t) & \equiv & x_0 \\ x_{k+1}(t) & = & x_0 + \int_{t_0}^{t} f(x, x_k(s))ds \quad \text{for} \quad k \geq 0 \end{cases}$$

The argument in the proof of the C.M.F.-P.T. gives only **uniform** estimates of the iteration error $(x_{k+1} - x_k)$, e.g.,

$$\|x_{k+1} - x_k\|_\infty \leq L\alpha\|x_k - x_{k-1}\|_\infty,$$

leading to the condition $\alpha < \frac{1}{L}$. For the Picard iteration, better results using **pointwise** estimates of $x_{k+1} - x_k$ are possible. The condition $\alpha < \frac{1}{L}$ turns out to be unnecessary. For the moment, we will set aside the uniqueness question and focus on existence.

Theorem: (Picard Global Existence for (IE) for Lipschitz f)
Let $I = [t_0, t_0 + \beta]$ for $0 < \beta$. Suppose $f(t, x)$ is in (C, Lip) on $I \times \mathbb{F}^n$. Then there is a solution $x_*(t)$ to the integral equation (IE) in $C(I)$.

Theorem: (Picard Local Existence for (IE) for Lipschitz f)
Let $I = [t_0, t_0 + \beta]$ for $0 < \beta$ and

$$\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq r\},$$

and suppose $f(t, x)$ is in (C, Lip) on $I \times \Omega$. Then there exists a solution $x_*(t)$ to the integral equation (IE) in $C(I_\alpha)$ where

$$I_\alpha = [t_0, t + \alpha], \quad \alpha = \min\left(\beta, \frac{r}{M}\right),$$

and where $M = \max_{(t,x) \in I \times \Omega} |f(t, x)|$.

We prove these two Theorems together.

Existence Proof

Let $X = C(I, \mathbb{F}^n)$ and $X = X_{\alpha,r} \equiv \{x \in C(I_\alpha) : \|x - x_0\|_\infty \le r\}$, for the global and local results, resp.ly. Then

$$(g(x))(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$$

maps X into X in both cases, and X is complete. Let

$$x_0(t) \equiv x_0 \quad \text{and} \quad x_{k+1} = g(x_k) \quad \text{for} \quad k \ge 0.$$

Let

$$M_0 = \max_{t \in I} |f(t, x_0)| \qquad \text{(global thm)},$$
$$M_0 = \max_{t \in I_\alpha} |f(t, x_0)| \qquad \text{(local thm)}.$$

Then for $t \in I$ (global) or $t \in I_\alpha$ (local),

$$\begin{aligned}
|x_1(t) - x_0| &\le \int_{t_0}^{t} |f(s, x_0)|\,ds \le M_0(t - t_0) \\
|x_2(t) - x_1(t)| &\le \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_0(s))|\,ds \\
&\le L \int_{t_0}^{t} |x_1(s) - x_0(s)|\,ds \\
&\le M_0 L \int_{t_0}^{t} (s - t_0)\,ds = \frac{M_0 L (t - t_0)^2}{2!}\ .
\end{aligned}$$

Existence Proof

By induction, suppose

$$|x_k(t) - x_{k-1}(t)| \le M_0 L^{k-1} \frac{(t-t_0)^k}{k!}.$$

Then

$$\begin{aligned}
|x_{k+1}(t) - x_k(t)| &\le \int_{t_0}^{t} |f(s, x_k(s)) - f(s, x_{k-1}(s))| ds \\
&\le L \int_{t_0}^{t} |x_k(s) - x_{k-1}(s)| ds \\
&\le M_0 L^k \int_{t_0}^{t} \frac{(s-t_0)^k}{k!} ds = M_0 L^k \frac{(t-t_0)^{k+1}}{(k+1)!}.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{k=0}^{\infty} |x_{k+1}(t) - x_k(t)| &\le \frac{M_0}{L} \sum_{k=0}^{\infty} \frac{(L(t-t_0))^{k+1}}{(k+1)!} \\
&= \frac{M_0}{L}(e^{L(t-t_0)} - 1) \le \frac{M_0}{L}(e^{L\gamma} - 1),
\end{aligned}$$

where $\gamma = \beta$ (global) or $\gamma = \alpha$ (local).

Existence Proof

Hence the series $x_0 + \sum_{k=0}^{\infty}(x_{k+1}(t) - x_k(t))$, which has x_{N+1} as its N^{th} partial sum, converges absolutely and uniformly on I (global) or I_α (local) by the Weierstrass M-test. Let

$$x_*(t) \in C(I) \quad \text{(global)} \quad \text{or} \quad x_*(t) \in C(I_\alpha) \quad \text{(local)}$$

be the limit function. Since

$$|f(t, x_k(t)) - f(t, x_*(t))| \leq L|x_k(t) - x_*(t)|,$$
$$f(t, x_k(t)) \longrightarrow f(t, x_*(t))$$

on I (global) or I_α (local). Therefore,

$$\begin{aligned} g(x_*)(t) &= x_0 + \int_{t_0}^{t} f(s, x_*(s))\,ds \\ &= \lim_{k\to\infty}(x_0 + \int_{t_0}^{t} f(s, x_k(x))\,ds) \\ &= \lim_{k\to\infty} x_{k+1}(t) = x_*(t), \end{aligned}$$

for all $t \in I$ (global) or I_α (local). Hence $x_*(t)$ is a fixed point of g in X, and thus also a solution of the integral equation (IE) in $C(I)$ (global) or $C(I_\alpha)$ (local.) $\square$

Existence Corollary

Corollary. The solution $x_*(t)$ of the integral equation (IE) satisfies

$$|x_*(t) - x_0| \leq \frac{M_0}{L}(e^{L(t-t_0)} - 1)$$

$$t \in I \quad \text{(global)} \quad \text{or} \quad t \in I_\alpha \quad \text{(local)},$$

where

$$M_0 = \max_{t \in I} |f(t, x_0)| \qquad \text{(global)},$$

$$M_0 = \max_{t = I_\alpha} |f(t, x_0)| \qquad \text{(local)}.$$

Remark. In each of the statements of the last three Theorems, we could replace

"solution of the integral equation (IE)"

with

"C^1 solution of the IVP: $DE : x' = f(t, x)$; $IC : x(t_0) = x_0$"

because of the equivalence of these two problems.

Example 1

(1) Consider a **linear** system

$$x' = A(t)x + b(t),$$

where

$$A(t) \in \mathbb{C}^{n \times n} \quad \text{and} \quad b(t) \in \mathbb{C}^n$$

are in $C(I)$ with $I = [t_0, t_0 + \beta]$.

Then f is in (C, Lip) on $I \times \mathbb{F}^n$

$$\begin{aligned} |f(t,x) - f(t,y)| &\leq |A(t)x - A(t)y| \\ &\leq \left(\max_{t \in I} \|A(t)\| \right) |x - y|. \end{aligned}$$

Hence there is a solution of the IVP

$$x' = A(t)x + b(t), \quad x(t_0) = x_0 \qquad \text{in} \quad C^1(I).$$

Example 2

(2) $(n = 1)$ Consider the IVP

$$x' = x^2, \quad x(0) = x_0 > 0.$$

Then $f(t, x) = x^2$ is not in (C, Lip) on $I \times \mathbb{R}$.
It is, however, in (C, Lip) on $I \times \Omega$ where

$$\Omega = \overline{B_r(x_0)} = [x_0 - r, x_0 + r]$$

for each fixed r. For a given $r > 0$, we have

$$M = (x_0 + r)^2, \quad \text{and} \quad \alpha = \frac{r}{M} = \frac{r}{(x_0 + r)^2}$$

in the local theorem. This value of α is maximized for $r = x_0$, where $\alpha = \frac{1}{4x_0}$. So the local theorem guarantees a solution in $\left[0, \frac{1}{4x_0}\right]$.
The actual solution $x_*(t) = (x_0^{-1} - t)^{-1}$ exists in $\left[0, \frac{1}{x_0}\right)$.

It is also possible to prove a local existence theorem assuming only that f is continuous, without assuming the Lipschitz condition. We need the following form of Ascoli's Theorem.

Definition: A sequence of functions $\{f_k\}$ between metric spaces X and Y is said to be **equicontinuous** if

$$(\forall\, \epsilon > 0)(\exists\, \delta > 0) \quad \text{such that} \quad (\forall\, k \geq 1)(\forall\, x_1, x_2 \in X)$$

$$d_x(x_1, x_2) < \delta \quad \Longrightarrow \quad d_Y(f_k(x_1), f_k(x_2)) < \epsilon\ .$$

Theorem: (Ascoli)
Let X and Y be metric spaces with X compact. Let $\{f_k\}$ be an **equicontinuous** sequence of functions $f_k : X \to Y$, and suppose for each $x \in X$,

$$\overline{\{f_k(x) : k \geq 1\}}$$

is a compact subset of Y. Then there is a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ and a continuous $f : X \to Y$ such that

$$f_{k_j} \to f \qquad \text{uniformly on } X.$$

Remarks on Ascoli's Theorem

(1) If $\{f_k\}$ is an **equicontinuous** sequence of functions between metric spaces, then each of the functions f_k must be continuous.

(2) If $Y = \mathbb{F}^n$, the condition

$$(\forall\, x \in X) \quad \overline{\{f_k(x) : k \geq 1\}}$$

is compact is equivalent to the sequence $\{f_k\}$ being **pointwise bounded**, i.e.,

$$(\forall\, x \in X)(\exists\, M_x) \quad \text{such that} \quad (\forall\, k \geq 1)$$
$$|f_k(x)| \leq M_x \ .$$

Remarks on Ascoli's Theorem

(3) Suppose $f_k : [a, b] \to \mathbb{R}$ is a sequence of C^1 functions, and suppose

$$\exists\, M \quad \text{such that} \quad (\forall\, k \geq 1)$$
$$\|f_k\|_\infty + \|f_k'\|_\infty \leq M \ ,$$

where

$$\|f\|_\infty = \max_{a \leq x \leq b} |f(x)| \ .$$

Then for $a \leq x_1 < x_2 \leq b$,

$$|f_k(x_2) - f_k(x_1)| \leq \int_{x_1}^{x_2} |f_k'(x)| dx \leq M|x_2 - x_1| \ ,$$

so $\{f_k\}$ is equicontinuous (take $\delta = \frac{\epsilon}{M}$), with $\|f_k\|_\infty \leq M$ implying that $\{f_k\}$ is pointwise bounded.
Thus, by Ascoli's Theorem, some subsequence of $\{f_k\}$ converges uniformly to a continuous function $f : [a, b] \to \mathbb{R}$.

The Cauchy-Peano Existence Theorem

Let $I = [t_0, t_0 + \beta]$ and

$$\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \le r\},$$

and suppose $f(t, x)$ is continuous on $I \times \Omega$.
Then there exists a solution $x_*(t)$ of the integral equation

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

in $C(I_\alpha)$ where $I_\alpha = [t_0, t_0 + \alpha]$,

$$\alpha = \min\left(\beta, \frac{r}{M}\right),$$

and

$$M = \max_{(t,x) \in I \times \Omega} |f(t, x)|,$$

and so $x_*(t)$ is a C^1 solution of the initial value problem

$$IVP: \qquad x' = f(t, x), \quad x(t_0) = x_0$$

in I_α.

Proof of the Cauchy-Peano Existence Theorem

The idea is to construct approximate solutions based on piecewise linear interpolants of grid functions generated by Euler's method, and then use Ascoli's Theorem to take the uniform limit of some subsequence. For each integer $k \geq 1$, define

$$x_k(t) \in C(I_\alpha)$$

as follows:

(a) partition $[t_0, t_0 + \alpha]$ into k equal subintervals, for

$$0 \leq \ell \leq k, \quad \text{let} \quad t_\ell = t_0 + \ell \frac{\alpha}{k} \ ,$$

(b) set $x_k(t_0) = x_0$, and

(c) for $\ell = 1, 2, \ldots, k$ define $x_k(t)$ in $(t_{\ell-1}, t_\ell]$ inductively by

$$x_k(t) = x_k(t_{\ell-1}) + f(t_{\ell-1}, x_k(t_{\ell-1}))(t - t_{\ell-1}).$$

Proof of the Cauchy-Peano Existence Theorem

Since f is only defined on $I \times \Omega$, we must check that

$$|x_k(t_{\ell-1}) - x_0| \leq r \quad \text{for} \quad 2 \leq \ell \leq k$$

for this to be well-defined. We show this by induction.
It is obvious for $\ell = 1$; inductively, we have

$$\begin{aligned}
|x_k(t_{\ell-1}) - x_0| &\leq \sum_{i=1}^{\ell-1} |x_k(t_i) - x_k(t_{i-1})| \\
&= \sum_{i=1}^{\ell-1} |f(t_{i-1}, x_k(t_{i-1}))| \cdot |t_i - t_{i-1}| \\
&\leq M \sum_{i=1}^{\ell-1} (t_i - t_{i-1}) \\
&= M(t_{\ell-1} - t_0) \\
&\leq M\alpha \leq r
\end{aligned}$$

by the choice of α. So $x_k(t) \in C(I_\alpha)$ is well defined.

Proof of the Cauchy-Peano Existence Theorem

A similar estimate shows that for $t, \tau \in [t_0, t_0 + \alpha]$,

$$|x_k(t) - x_k(\tau)| \le M|t - \tau| \ .$$

This implies that $\{x_k\}$ is equicontinuous; it also implies that

$$(\forall\, k \ge 1)(\forall\, t \in I_\alpha) \quad |x_k(t) - x_0| \le M\alpha \ \le r,$$

so $\{x_k\}$ is pointwise bounded (in fact, uniformly bounded).

Proof of the Cauchy-Peano Existence Theorem

Now, by Ascoli's Theorem, some subsequence

$$\{x_{k_j}\}_{j=1}^{\infty}$$

converges uniformly to some

$$x_*(t) \in C(I_\alpha).$$

It remains to show that $x_*(t)$ is a solution of (IE) on I_α.
Since each $x_k(t)$ is continuous and piecewise linear on I_α,

$$x_k(t) = x_0 + \int_{t_0}^{t} x_k'(s)\,ds,$$

where $x_k'(t)$ is piecewise constant on I_α and is defined for all t except

$$t_\ell \qquad (1 \le \ell \le k-1).$$

At each $t_\ell \qquad (1 \le \ell \le k-1)$, set

$$x_k'(t) = \lim_{t \downarrow t_\ell} x_k'(t) = f(t_\ell, x_k(t_\ell))$$

$$\Delta_k(t) = x_k'(t) - f(t, x_k(t)) \quad \text{on} \quad I_\alpha$$

and note that $\Delta_k(t_\ell) = 0$ for $0 \le \ell \le k-1$ by definition.

Proof of the Cauchy-Peano Existence Theorem

Claim: $\Delta_k(t) \to 0$ uniformly on I_α as $k \to \infty$.
Proof of Claim: Given k, for

$$1 \le \ell \le k \quad \text{and} \quad t \in [t_{\ell-1}, t_\ell),$$
$$\text{including } t_k \text{ if } \ell = k,$$
$$\Delta_k(t) = |x_k'(t) - f(t, x_k(t))| = |f(t_{\ell-1}, x_k(t_{\ell-1})) - f(t, x_k(t))|.$$

Noting that

$$|t - t_{\ell-1}| \le \frac{\alpha}{k}$$

and

$$|x_k(t) - x_k(t_{\ell-1})| \le M|t - t_{\ell-1}| \le M\frac{\alpha}{k},$$

the uniform continuity of f on the compact set $I \times \Omega$ implies

$$\max_{t \in I_\alpha} |\Delta_k(t)| \to 0 \quad \text{as} \quad k \to \infty\ .$$

So, in particular,

$$\Delta_{k_j}(t) \to 0 \quad \text{uniformly on} \quad I_\alpha.$$

□

Proof of the Cauchy-Peano Existence Theorem

Now

$$(*) \qquad x_{k_j}(t) = x_0 + \int_{t_0}^{t} x'_{k_j}(s)\,ds$$
$$= x_0 + \int_{t_0}^{t} f(s, x_{k_j}(s))\,ds + \int_{t_0}^{t} \Delta_{k_j}(s)\,ds.$$

Since $x_{k_j} \to x_*$ uniformly on I_α, the uniform continuity of f on $I \times \Omega$ implies that

$$f(t, x_{k_j}(t)) \to f(t, x_*(t)) \quad \text{uniformly on} \quad I_\alpha.$$

Taking the limit as $j \to \infty$ on both sides of $(*)$ for each $t \in I_\alpha$, we obtain that x_* satisfies (IE) on I_α □