Linear Analysis
Lecture 20

An ODE is an equation of the form

$$g(t, x, x', \ldots, x^{(m)}) = 0, \text{ where } g : \Omega \subset \mathbb{R} \times (\mathbb{F}^n)^{m+1} \mapsto \mathbb{F}^n.$$

A **solution** on an interval $I \subset \mathbb{R}$ is a function $x : I \to \mathbb{F}^n$ for which

$$x'(t), x''(t), \ldots, x^{(m)}(t) \text{ exists on } I \text{ and } g(t, x(t), x'(t), \ldots, x^{(m)}(t)) = 0 \; \forall \, t \in I.$$

We focus on the case where $x^{(m)}$ can be solved for explicitly:

$$x^{(m)} = f(t, x, x', \ldots, x^{(m-1)}), \text{ where}$$

$$f : D \subset \mathbb{R} \times (\mathbb{F}^n)^m \mapsto \mathbb{F}^n \qquad \text{is continuous.}$$

This equation is called an m^{th}**-order** $n \times n$ system of ODE's.

Note that if x is a solution defined on an interval $I \subset \mathbb{R}$, then the existence of $x^{(m)}$ on I (including one-sided limits at the endpoints of I) implies that $x \in C^{m-1}(I)$. Hence $x^{(m)} \in C(I)$ since f is continuous, so $x \in C^m(I)$.

Reduction to First-Order Systems

Every m^{th}-order $n \times n$ system of ODE's is equivalent to a first-order $mn \times mn$ system of ODE's.

Define

$$y_j(t) = x^{(j-1)}(t) \in \mathbb{F}^n \qquad 1 \le j \le m \text{ and}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \in \mathbb{F}^{mn},$$

the system

$$x^{(m)} = f(t, x, \ldots, x^{(m-1)})$$

is equivalent to the first-order $mn \times mn$ system

$$y' = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(t, y_1, \ldots, y_m) \end{bmatrix}.$$

By relabeling we can focus on first-order $n \times n$ systems of the form

$$x' = f(t, x),$$

where $f : \mathbb{R} \times \mathbb{F}^n \mapsto \mathbb{F}^n$ is continuous.

Example of a first-order system

Consider $x'(t) = f(t)$ where $f : I \to \mathbb{F}^n$ is continuous on $I \subset \mathbb{R}$. For a fixed $t_0 \in I$, the general solution of the ODE is

$$x(t) = c + \int_{t_0}^{t} f(s)ds,$$

where $c \in \mathbb{F}^n$ is an arbitrary.

Initial-Value Problems (IVP's) for First-order Systems

Under certain conditions on f, the general solution of a first-order system $x' = f(t, x)$ involves n arbitrary constants in $\mathbb{F}$.

So n scalar conditions must be given to specify a particular solution.

For the example above, clearly giving $x(t_0) = x_0$ determines c.

An IVP for the first-order system is the differential equation

$$DE : \quad x' = f(t, x),$$

together with initial conditions

$$IC : \quad x(t_0) = x_0.$$

A **solution** of the IVP is a solution $x(t)$ of the DE, defined on an interval I containing t_0, which also satisfies the IC.

Examples

(1) Let $n = 1$.

$$IVP: \quad \begin{cases} DE: & x' = x^2 \\ IC: & x(1) = 1 \end{cases}$$

is

$$x(t) = \frac{1}{2-t},$$

which blows up as $t \to 2$. So even if f is C^∞ on all of $\mathbb{R} \times \mathbb{F}^n$, solutions of an IVP do not necessarily exist for all time t.

(2) Let $n = 1$.

$$IVP: \quad \begin{cases} DE: & x' = 2\sqrt{|x|} \\ IC: & x(0) = 0 \ . \end{cases}$$

For any $c \geq 0$, define

$$x_c(t) = \begin{cases} 0, & \text{for} \quad t \leq c, \\ (t-c)^2, & \text{for} \quad t \geq c. \end{cases}$$

Then, for every $c \geq 0$, $x_c(t)$ is a solution of this IVP.
So, in general, for continuous $f(t, x)$, IVP's may have non-unique solutions.
The difficulty here is that $f(t, x) = 2\sqrt{|x|}$ does not satisfy a Lipschitz condition in x near $x = 0$.

An Integral Equation Equivalent to an IVP

Suppose $x(t) \in C^1(I)$ is a solution of

$$\begin{cases} DE: & x' = f(t, x) \\ IC: & x(t_0) = x_0 \end{cases} \tag{IVP}$$

on the interval $I \subset \mathbb{R}$ with $t_0 \in I$, where f is continuous. Then $\forall\ t \in I$,

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\,ds = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds,$$

so $x(t)$ is also a solution of the **integral equation**

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds \qquad (t \in I). \tag{IE}$$

Conversely, if $x(t) \in C(I)$ is a solution of (IE), then $f(t, x(t)) \in C(I)$, so

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds \in C^1(I)$$

and $x'(t) = f(t, x(t))$ by the Fundamental Theorem of Calculus. So x is a C^1 solution of the DE on I, and $x(t_0) = x_0$, so x is a solution of (IVP).

An Integral Equation Equivalent to an IVP

Proposition. On an interval I containing t_0, x is a solution of the initial value problem (IVP) with $x \in C^1(I)$ iff x is a solution of the integral equation (IE) on I with $x \in C(I)$.

The Contraction Mapping Fixed-Point Theorem

The integral equation (IE) transforms the initial value problem (IVP) to a problem on $C(I)$ without concern for differentiability. Moreover, the initial condition is built into the integral equation.

We solve (IE) using a fixed-point formulation.

Definition. Let (X, d) be a metric space, and suppose $g : X \to X$. We say that g is a **contraction** if

$$\exists\, c < 1 \quad \text{such that} \quad d(g(x), g(y)) \leq c d(x, y) \quad \forall\, x, y \in X.$$

A point $x_* \in X$ for which $g(x_*) = x_*$ is called a **fixed point** of g.

A contraction is a Lipschitz continuous function with Lipschitz constant < 1.

The Contraction Mapping Fixed-Point Theorem

The Contraction Mapping Fixed-Point Theorem

Let (X, d) be a **complete** metric space and

$$g : X \to X$$

a contraction (with contraction constant $c < 1$).

Then g has a unique fixed point $x_* \in X$.

Moreover, for $x_0 \in X$, if $\{x_k\}$ is generated by the **fixed point iteration**

$$x_{k+1} = g(x_k) \quad \text{for} \quad k \geq 0,$$

then $x_k \to x_*$.

Proof of the Contraction Mapping Fixed-Point Theorem

Fix $x_0 \in X$, and set $x_{k+1} = g(x_k) \quad \text{for} \quad k \geq 0$, Then for $k \geq 1$,

$$d(x_{k+1}, x_k) = d(g(x_k), g(x_{k-1})) \leq cd(x_k, x_{k-1}).$$

By induction, $d(x_{k+1}, x_k) \leq c^k d(x_1, x_0)$. So for $n < m$,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq \left(\sum_{j=n}^{m-1} c^j \right) d(x_1, x_0) \\ &\leq \left(\sum_{j=n}^{\infty} c^j \right) d(x_1, x_0) = \frac{c^n}{1-c} d(x_1, x_0). \end{aligned}$$

Since $c^n \to 0$ as $n \to \infty$, $\{x_k\}$ is Cauchy. Since X is complete, $x_k \to x_*$ for some $x_k \in X$. Since g is continuous,

$$g(x_*) = g(\lim x_k) = \lim g(x_k) = \lim x_{k+1} = x_*,$$

so x_* is a fixed point.

If x and y are two fixed points of g in X, then

$$d(x, y) = d(g(x), g(y)) \leq cd(x, y),$$

so $(1-c)d(x, y) \leq 0$. Thus and $x = y$. So g has a unique fixed point.

Uniformly Lipschitz Continuity

Applying the Contraction Mapping Fixed-Point Theorem (C.M.F.-P.T.) to a function usually requires two steps:
(i) showing there is a complete set S for which $g(S) \subset S$, and
(ii) showing that g is a contraction on S.

To apply the C.M.F.-P.T. to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds, \qquad (IE)$$

we need a further condition on f.

Definition. Let $|\cdot|$ be the Euclidean norm on $\mathbb{F}^n$. Let $I \subset \mathbb{R}$ be an interval and $\Omega \subset \mathbb{F}^n$. The function $f : I \times \Omega \mapsto \mathbb{F}^n$ is **uniformly Lipschitz continuous with respect to** x if

$$|f(t,x) - f(t,y)| \leq L|x-y| \quad (\forall\, t \in I)(\forall\, x, y \in \Omega).$$

We say that f is in (C, Lip) on $I \times \Omega$ if f is continuous on $I \times \Omega$ and f is uniformly Lipschitz continuous with respect to x on $I \times \Omega$.

For simplicity, we will consider intervals $I \subset \mathbb{R}$ for which t_0 is the left endpoint. Virtually identical arguments hold if t_0 is the right endpoint of I, or if t_0 is in the interior of I.

Local Existence and Uniqueness for (IVP)

Theorem: Let $\beta > 0$, $\hat{r} > 0$, and define

$$I = [t_0, t_0 + \beta] \quad \text{and} \quad \Omega = \overline{B_{\hat{r}}(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq \hat{r}\},$$

Suppose $f(t, x)$ is in (C, Lip) on $I \times \Omega$. Then there exists $0 < \alpha \leq \beta$ for which there is a unique solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \tag{IE}$$

in $C(I_\alpha)$ where $I_\alpha = [t_0, t_0 + \alpha]$.

Moreover, we can choose $\alpha \in (0, \beta]$ to be any positive number satisfying

$$\alpha \leq \frac{\hat{r}}{M} \quad \text{and} \quad \alpha < \frac{1}{L}, \quad \text{where} \quad M = \max_{(t,x) \in I \times \Omega} |f(t, x)|$$

and L is the Lipschitz constant for f in $I \times \Omega$.

Proof of Local Existence and Uniqueness for (IVP)

For any $\alpha \in (0, \beta]$, let $\|\cdot\|_\infty$ denote the max-norm on $C(I_\alpha)$ (i.e. the uniform convergence norm). Then $(C(I_\alpha), \|\cdot\|_\infty)$ is a Banach space.

Let $\widetilde{x}_0$ denote the constant function $\widetilde{x}_0(t) \equiv x_0$ in $C(I_\alpha)$. Define

$$X_{\alpha,r} = \{x \in C(I_\alpha) : \|x - \widetilde{x}_0\|_\infty \leq r\}.$$

Then $X_{\alpha,r}$ is a complete metric space since it is a closed subset of the Banach space $(C(I_\alpha), \|\cdot\|_\infty)$. Define $g : X_{\alpha,r} \to C(I_\alpha)$ by

$$(g(x))(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \ .$$

The mapping g is well-defined on $X_{\alpha,r}$ and $g(x) \in C(I_\alpha)$ for $x \in X_{\alpha,r}$ since f is continuous on $I \times \overline{B_r(x_0)}$. Fixed points of g are solutions of the integral equation (IE).

Proof of Local Existence and Uniqueness for (IVP)

Claim Suppose $\alpha \in (0, \beta]$ and $\alpha \leq \min\left\{\frac{r}{M}, \ \frac{1}{L}\right\}$. Then g maps $X_{\alpha,r}$ into itself and g is a contraction on $X_{\alpha,r}$ with contraction coefficient αL.
Proof: If $x \in X_{\alpha,r}$, then for $t \in I_\alpha$,

$$|(g(x))(t) - x_0| \leq \int_{t_0}^{t} |f(s, x(s))| ds \leq M\alpha \leq r,$$

so $g : X_{\alpha,r} \to X_{\alpha,r}$. If $x, y \in X_{\alpha,r}$, then for $t \in I_\alpha$,

$$\begin{aligned} |(g(x))(t) - (g(y))(t)| &\leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| ds \\ &\leq \int_{t_0}^{t} L|x(s) - y(s)| ds \\ &\leq L\alpha \|x - y\|_\infty, \end{aligned}$$

so

$$\|g(x) - g(y)\|_\infty \leq L\alpha \|x - y\|_\infty,$$

where $L\alpha < 1$, that is g is a contraction on $X_{\alpha,r}$. $\square$

Proof of Local Existence and Uniqueness for (IVP)

By the C.M.F.-P.T., g has a unique fixed point in $X_{\alpha,r}$. Thus the integral equation (IE) has a unique solution $x_*(t)$ in

$$X_{\alpha,r} = \{x \in C(I_\alpha) : \|x - \widetilde{x}_0\|_\infty \le r\}.$$

We now show uniqueness.
Fix $\alpha > 0$. For $0 < \gamma \le \alpha$, $x_*|_{I_\gamma}$ is the unique fixed point of g on $X_{\gamma,r}$.
Suppose $y \in C(I_\alpha)$ is a solution of (IE) on I_α with $y \not\equiv x_\alpha$ on I_α. Let

$$\gamma_1 = \inf\{\gamma \in (0,\alpha] : y(t_0+\gamma) \ne x_*(t_0+\gamma)\}.$$

By continuity, $\gamma_1 < \alpha$. Since $y(t_0) = x_0$, continuity implies

$$\exists\, \gamma_0 \in (0,\alpha] \quad \text{such that} \quad y|_{I_{\gamma_0}} \in X_{\gamma_0,r}.$$

Thus $y(t) \equiv x_*(t)$ on I_{γ_0}.

Proof of Local Existence and Uniqueness for (IVP)

So $0 < \gamma_1 < \alpha$. Since $y(t) \equiv x_*(t)$ on I_{γ_1}, $y|_{I_{\gamma_1}} \in X_{\gamma_1,r}$. Let $\rho = M\gamma_1$, then $\rho < M\alpha \leq r$. For $t \in I_{\gamma_1}$,

$$|y(t) - x_0| = |(g(y))(t) - x_0| \leq \int_{t_0}^{t} |f(s, y(s))| ds \leq M\gamma_1 = \rho,$$

so $y|_{I_{\gamma_1}} \in X_{\gamma_1,\rho}$. By continuity,

$$\exists\, \gamma_2 \in (\gamma_1, \alpha] \quad \text{such that} \quad y\Big|_{I_{\gamma_2}} \in X_{\gamma_1,r}.$$

But then $y(t) \equiv x_*(t)$ on I_{γ_2}, contradicting the definition of γ_1. □