Linear Analysis
Lecture 2
Dual Vector Spaces

Dual Vector Spaces

Let V be a vector space.
A **linear functional** on V is a function $f : V \to \mathbb{F}$ for which

$$f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2) \quad \forall\ v_1, v_2 \in V,\ \alpha_1, \alpha_2 \in \mathbb{F}.$$

- If V is a finite-dimensional vector space, the dual space of V is the vector space V^* of all linear functionals on V.
- When V is infinite dimensional, the set of **all** linear functions is often called the **algebraic** dual space of V, as it depends only on the algebraic structure of V. We will be more interested in linear functionals related to a topological structure on V.

Dual Vector Spaces: $\mathbb{F}^n$

$$
\begin{aligned}
V &= \mathbb{F}^n \\
f &- \text{a linear functional on } V \\
f_i &= f(e_i) \text{ for } 1 \le i \le n
\end{aligned}
$$

For $x \in \mathbb{F}^n$, $x = (x_1, \ldots, x_n)^T = \sum_{i=1}^n x_i e_i \in \mathbb{F}^n$, and

$$
\begin{aligned}
f(x) &= \sum_{i=1}^n x_i f(e_i) \\
&= \sum_{i=1}^n f_i x_i = (f_1 \; f_2 \; \cdots \; f_n)\, x.
\end{aligned}
$$

So the row vector $(f_1 \cdots f_n)$ is the matrix of f using the standard basis $\{e_1, \ldots, e_n\}$ on $\mathbb{F}^n$ (and the basis $\{1\}$ on $\mathbb{F}$).
Hence, $(\mathbb{F}^n)^*$ is isomorphic to $\mathbb{F}^n$.

Dual Vector Spaces: $\ell^1(\mathbb{F})$

Let $V = \ell^1(\mathbb{F})$. If

$$f \in \ell^\infty(\mathbb{F}),$$

then for $x \in \ell^1(\mathbb{F})$,

$$\sum_{i=1}^{\infty} |f_i x_i| \leq (\sup |f_i|) \sum_{i=1}^{\infty} |x_i| < \infty,$$

so the sum

$$f(x) = \sum_{i=1}^{\infty} f_i x_i$$

converges absolutely, defining a linear functional on $\ell^1(\mathbb{F})$.

Dual Vector Spaces: $\ell^\infty(\mathbb{F})$

Similarly, if

$$V = \ell^\infty(\mathbb{F})$$

and $f \in \ell^1(\mathbb{F})$,

$$f(x) = \sum_{i=1}^{\infty} f_i x_i$$

defines a linear functional on $\ell^\infty(\mathbb{F})$.

Dual Vector Spaces: $C^1(X)$

Let $X \subset \mathbb{R}^n$, $x_0 \in X$, $u \in C^1(X)$.

- The function
$$f(u) = u(x_0)$$
defines a linear functional on $C^1(X)$.
- The function
$$f(u) = u'(x_0)$$
defines a linear functional on $C^1(X)$.
- If $-\infty < a < b < \infty$ and $X = [a, b]$, the function
$$f(u) = \int_a^b u(x)dx$$
defines a linear functional on $C([a, b])$.

Dual Basis in Finite Dimensions

V is a finite dimensional vector space and $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V. For $1 \leq i \leq n$, define linear functionals $f_i \in V^*$ by

$$f_i(v_j) = \delta_{ij} \quad (= 1 \text{ for } i = j, \quad = 0 \text{ for } i \neq j).$$

Let $v \in V$, and let $x = (x_1, \ldots, x_n)^T$ be the vector of coordinates of v w.r.t $\mathcal{B}$ i.e., $v = \sum_{i=1}^n x_i v_i$.
Then

$$f_i(v) = f_i(\sum_{j=1}^n x_j v_j) = \sum_{j=1}^n x_j f_i(v_j) = x_i,$$

i.e., f_i maps v into its coordinate x_i of v_i.
Now if $f \in V^*$, let $a_i = f(v_i)$ $(1 \leq i \leq n)$; then

$$f(v) = f(\textstyle\sum_{i=1}^n x_i v_i) \;=\; \sum_{i=1}^n x_i f(v_i) = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i f_i(v),$$

so $f = \sum_{i=1}^n a_i f_i$.

Dual Basis in Finite Dimensions

Since this representation is unique, $\{f_1, \ldots, f_n\}$ is a basis for V^*, called the **dual basis** to $\{v_1, \ldots, v_n\}$.

We get $\dim V^* = \dim V$.

Formally it is useful to think of the elements $f \in V^*$ as "row" vectors. With this formal interpretation, write the dual basis as a column of "row" vectors

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} .$$

Also write the coordinates of f w.r.t. the dual basis as a row vector $(a_1 \cdots a_n)$, where $f = \sum_{i=1}^n a_i f_i$. Then

$$f = \sum_{i=1}^n a_i f_i = (a_1 \cdots a_n) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} .$$

Dual Basis in Finite Dimensions

The defining equation of the dual basis becomes

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (v_1 \cdots v_n) = \begin{pmatrix} 1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & 1 \end{pmatrix} = I \qquad (*)$$

Change of Basis and Dual Bases

Let $\mathcal{B}_1 = \{w_1, \ldots, w_n\}$ and $\mathcal{B}_2 = \{v_1, \ldots, v_n\}$ be bases of V with change-of-basis matrix A, i.e., $(w_1 \cdots w_n) = (v_1 \cdots v_n)A$.
Left and right multiply $(*)$ by A^{-1} and A reps.ly

$$A^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (v_1 \cdots v_n)A = A^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (w_1 \cdots w_n) = I.$$

Therefore,

$$\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = A^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \quad \text{satisfies} \quad \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} (w_1 \cdots w_n) = I$$

so $\{g_1, \ldots, g_n\}$ is the dual basis to $\{w_1, \ldots, w_n\}$.

Change of Basis and Dual Bases

Let $f \in V^*$ have

$$\begin{array}{lll} \{f_1, \ldots, f_n\} & \text{coordinates} & (a_1 \cdots a_n) \quad \text{and} \\ \{g_1, \ldots, g_n\} & \text{coordinates} & (b_1 \cdots b_n) \end{array}$$

Then

$$f = (b_1 \cdots b_n) \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = (b_1 \cdots b_n) A^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = (a_1 \cdots a_n) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

so $(b_1 \cdots b_n) A^{-1} = (a_1 \cdots a_n)$, i.e.,

$$(b_1 \cdots b_n) = (a_1 \cdots a_n) A,$$

is the coordinate transformation law for these bases.

Linear Transformations from $\mathbb{F}^n$ to $\mathbb{F}^m$

$T : \mathbb{F}^n \to \mathbb{F}^m$ a linear transformation.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n, \quad \text{and} \quad \{e_j\} \quad \text{the standard basis}$$

Define

$$T(e_j) = T_{\cdot j} = \sum_{i=1}^m t_{ij} e_i = \begin{pmatrix} t_{ij} \\ \vdots \\ t_{mj} \end{pmatrix} \in \mathbb{F}^m \quad 1 \le j \le n$$

Then

$$\begin{aligned} T(x) &= T\left(\sum_{j=1}^n x_j e_j\right) = \sum_{j=1}^n x_j T(e_j) \\ &= (T_{\cdot 1} \cdots T_{\cdot n}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \cdots & t_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \end{aligned}$$

So every linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$ can be represented as multiplication by a matrix in $\mathbb{F}^{m \times n}$.

Some Special Elementary Matrices: Diagonal Matrices

$$D = \operatorname{diag}(d_1, d_2, \dots, d_n) \text{ where } D_{ij} = \begin{cases} d_i, & \text{if } i = j \\ 0, & \text{else.} \end{cases}$$

The identity matrix is diagonal with $I = \operatorname{diag}(1, 1, \dots, 1)$.
The scalar multiple of I is called a scalar matrix.

Left multiplication of $A \in \mathbb{F}^{n \times m}$ by the diagonal matrix D *rescales the rows* of the matrix,

$$(DA)_{i\cdot} = \delta_i A_{i\cdot} \qquad i = 1, \dots, n.$$

Right multiplication of $B \in \mathbb{F}^{m \times n}$ rescales the columns,

$$(BD)_{\cdot j} = \delta_j B_{\cdot j} \qquad j = 1, \dots, n.$$

Shift Matrices

Forward Shift: $S \in \mathbb{F}^{n \times n}$ $\qquad s_{ij} = \delta_{i(j+1)}$

$$S = S_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad S \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{pmatrix}$$

Backward Shift: $B \in \mathbb{C}^{n \times n}$ $\qquad b_{ij} \in \delta_{(i+1)j}$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$S^n = 0 = B^n$$

Circulant Matrices (Forward)

$$\begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_n & a_1 \end{bmatrix}$$

Basic circulant permutation matrices:
Forward:

$$C_f = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad C_f \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x_1 \end{pmatrix}$$

Backward:

$$C_b = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad C_b \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

More Matrices

Jordan Block: $J \in \mathbb{C}^{n \times n}$

$$J = \lambda I + S_n = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & 1 \\ 0 & 0 & & & & \lambda \end{bmatrix} \qquad \text{where } S_1 = 0.$$

Vandermonde Matrices: $x_1, \ldots, x_n \in \mathbb{F}$

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ \vdots & & & & & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{n-1} \end{bmatrix}$$

Vandermonde determinant $\det V = \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)$

Elementary Matrices

$E_{rs} \in \mathbb{F}^{m\times n}$ have a 1 is the (r, s)-entry and 0 elsewhere, e.g. in $\mathbb{F}^{4\times 5}$

$$E_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note $C_f = E_{n1} + S_n$.

The elementary matrices form the *standard basis* for $\mathbb{F}^{m\times n}$.

Multiplication by Elementary Matrices

Left multiplication of $T \in \mathbb{F}^{m \times n}$ by $E_{rs} \in \mathbb{F}^{m \times m}$ moves the sth row of T to the rth row and zeros out all other elements.

That is, the elements of the matrix $E_{rs}T$ are all zero except for those in the rth row which is just the sth row of T.

Right multiplication of $T \in \mathbb{F}^{m \times n}$ by $E_{rs} \in \mathbb{F}^{n \times n}$ moves the rth column of T to the sth column and zeros out all other elements.

That is, the elements of the matrix TE_{rs} are all zero except for those in the sth column which is just the rth column of T.