Linear Analysis
Lecture 19

Spectral Representation Theorem

Since $X = M_1 \oplus \cdots \oplus M_s$, we can write L as the sum of the operators $P_j L P_j$ $j = 1, \ldots, s$. Note that for each $j = 1, \ldots, s$

$$P_j L P_j = L P_j = \lambda_j P_j + N_j.$$

Therefore, $L = S + N$ where $S = \sum_{j=1}^{s} \lambda_j P_j$ and $N = \sum_{j=1}^{s} N_j$, with

$$\begin{aligned} I &= \sum_{j=1}^{s} P_j \\ P_k P_j &= P_j P_k = \delta_{jk} P_j \\ N_j P_k &= P_k N_j = \delta_{kj} N_j \\ N_j P_k &= P_k N_j = 0 \quad i \neq j \;. \end{aligned}$$

This representation is unique in the sense that any other such representation $L = S' + N'$ has $S = S'$ and $N = N'$.

Proof of the Spectral Representation Theorem

The proof makes use of the expansion of $R(\zeta)$ at ∞,

$$R(\zeta) = -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n$$

and its partial fractions decomposition,

$$R(\zeta) = -\sum_{j=1}^{s} \left[(\zeta - \lambda_j)^{-1} P_j + \sum_{n=1}^{m_j - 1} (\zeta - \lambda_j)^{-(n+1)} N_j^n \right] .$$

Γ a simple closed curve about the origin with $\Sigma(L)$ in its interior,

$$\begin{aligned}
L &= \frac{1}{2\pi i} \oint_\Gamma \frac{1}{\zeta} d\zeta \ L = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_\Gamma \zeta^{-n} d\zeta \ L^n = \frac{1}{2\pi i} \oint_\Gamma \sum_{n=0}^{\infty} \zeta^{-n} L^n d\zeta \\
&= \frac{1}{2\pi i} \oint_\Gamma \zeta \sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n d\zeta = -\frac{1}{2\pi i} \oint_\Gamma \zeta R(\zeta) d\zeta \\
&= \frac{1}{2\pi i} \oint_\Gamma \zeta \sum_{j=1}^{s} \left[(\zeta - \lambda_j)^{-1} P_j + \sum_{n=1}^{m_j - 1} (\zeta - \lambda_j)^{-(n+1)} N_j^n \right] d\zeta \\
&= \sum_{j=1}^{s} \left[\frac{1}{2\pi i} \oint_\Gamma \zeta (\zeta - \lambda_j)^{-1} d\zeta P_j + \sum_{n=1}^{m_j - 1} \frac{1}{2\pi i} \oint_\Gamma \zeta (\zeta - \lambda_j)^{-(n+1)} d\zeta N_j^n \right] \\
&= \sum_{j=1}^{s} \lambda_j P_j + N_j \ .
\end{aligned}$$

Terminology

$\Sigma(L) = \{\lambda_1, \ldots, \lambda_k\}$ with algebraic multiplicities $m_1, m_2, \ldots, m_k$.

$$
\begin{array}{lcl}
E_{\lambda_j}(L) & = & \ker(L - \lambda_j) \text{ the eigenspace for } \lambda_j \\
\hat{E}_{\lambda_j}(L) & = & \ker(L - \lambda_j)^{m_j} \text{ the generalized eigenspace for } \lambda_j \\
\dim(\hat{E}_{\lambda_j}(L)) & = & m_j \text{ the algebraic multiplicity of } \lambda_j. \\
P_j & = & \text{eigenprojection for the eigenvalue } \lambda_i. \\
 & = & \text{the projection onto } \hat{E}_{\lambda_j}(L) \text{ along } \bigoplus_{i \neq j} \hat{E}_{\lambda_i}(L). \\
N_j & = & \text{e-nilpotent for the eigenvalue } \lambda_j. \\
u \in \hat{E}_{\lambda_j}(L) \backslash \{0\} & = & \text{generalized eigenvectors of } \lambda_j
\end{array}
$$

Given $\lambda_j \in \Sigma(L)$,

λ_j is said to be simple if $m_j = 1$ $(N_j = 0)$.
λ_j is said to be semi-simple if $N_j = 0$.
λ_j is said to be degenerate if λ_j is not simple.
λ_j is said to be defective if $m_j > \dim\ker(L - \lambda_j I)$.
λ_j is said to be non-derogatory if $\operatorname{rank}(N_j) = m_j$.
λ_j is said to be derogatory if $\operatorname{rank}(N_j) < m_j$.

The Cauchy Integral Formula for Operators

Let $\phi(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots$ be a power series with radius of convergence $r > 0$. Then ϕ is holomorphic on the disc $|\zeta| < r$. If $L \in \mathcal{B}(V)$ has $\|L\| < r$, then as we have seen the operator

$$\phi(L) = a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \ldots$$

is well defined with the series being absolutely convergent.

Theorem: The mapping $\phi : \{T \in \mathcal{B}(V) : \|T\| < r\} \to \mathcal{B}(V)$ defined above satisfies the Cauchy integral formula. That is

$$\phi(L) = -\frac{1}{2\pi i}\oint_\Gamma \phi(\zeta) R(\zeta, L)\, d\zeta = \frac{1}{2\pi i}\oint_\Gamma \phi(\zeta)(\zeta - L)^{-1}\, d\zeta,$$

where Γ is any simple closed curve contained within the disc of radius r and $R(\zeta, L)$ is the resolvent for L.

Proof of the Cauchy Integral Formula for Operators

Recall that $\rho(L) \le \|L\|$, so for $\|L\| < |\zeta|$,

$$\begin{aligned} R(\zeta) &= -\zeta^{-1}(1-\zeta^{-1}L)^{-1} \\ &= -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n \ . \end{aligned}$$

Hence,

$$\begin{aligned} &-\phi(\zeta)R(\zeta) = \left(a_0 + a_1\zeta + a_2\zeta^2 + \dots\right)\left(\frac{1}{\zeta} + \frac{L}{\zeta^2} + \frac{L^2}{\zeta^3} + \dots\right) \\ &= \frac{1}{\zeta}\left(a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \dots\right) + \left(a_1 + a_2 L + a_3 L^2 + \dots\right) \\ &+ \frac{L}{\zeta^2}\left(a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \dots\right) + \zeta\left(a_2 + a_3 L + a_4 L^2 + \dots\right) \\ &+ \frac{L^2}{\zeta^2}\left(a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \dots\right) + \zeta^2\left(a_3 + a_4 L + a_5 L^2 + \dots\right) \end{aligned}$$

Hence, by the Residue Theorem (and uniform convergence),

$$\begin{aligned} -\frac{1}{2\pi i}\oint_\Gamma \phi(\zeta)R(\zeta, L)d\zeta &= \left(a_0 + a_1 L + a_2 L^2 + a_3 L^3 + \dots\right) \\ &= \phi(L) \ . \end{aligned}$$

Proposition. Suppose $L \in \mathcal{L}(V)$ and φ_1 and φ_2 are both holomorphic in a neighborhood of $\Sigma(L)$. Then
(a) $(a_1\varphi_1 + a_2\varphi_2)(L) = a_1\varphi_1(L) + a_2\varphi_2(L)$, and
(b) $(\varphi_1\varphi_2)(L) = \varphi_1(L) \circ \varphi_2(L)$.

Proof:
(a) follows from the linearity of contour integration.

To see (b) let Ω be the domain on which both φ_1 and φ_2 are holomorphic and which contains $\Sigma(L)$.
Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of L, with algebraic multiplicities $m_1, \ldots, m_k$, respectively.

For $s = 1, 2,\ j = 1, 2, \ldots, k$, let $\Gamma_{sj},\ s = 1, 2$, be two circles around λ_j with the radius of Γ_{2j} greater than that of Γ_{1j} and such that the discs $\Delta_{sj},\ s = 1, 2$, associated with Γ_{sj}, resp.ly, are contained in Ω.

Set $\Gamma_s = \bigcup_{j=1}^{k} \Gamma_{sj},\ s = 1, 2$.

By the first resolvent equation we get

$$
\begin{aligned}
&\varphi_1(L) \circ \varphi_2(L) \\
&= \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \varphi_1(\zeta_1) R(\zeta_1)\, d\zeta_1 \circ \oint_{\Gamma_2} \varphi_2(\zeta_2) R(\zeta_2)\, d\zeta_2 \\
&= \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \varphi(\zeta_1)\varphi(\zeta_2) R(\zeta_1) \circ R(\zeta_2)\, d\zeta_2\, d\zeta_1 \\
&= \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \varphi(\zeta_1)\varphi(\zeta_2) \frac{R(\zeta_1) - R(\zeta_2)}{\zeta_1 - \zeta_2}\, d\zeta_2\, d\zeta_1 \\
&= \frac{1}{(2\pi i)^2} \left[\oint_{\Gamma_1} \varphi_1(\zeta_1) R(\zeta_1) \oint_{\Gamma_2} \frac{\varphi_2(\zeta_2)}{\zeta_1 - \zeta_2}\, d\zeta_2\, d\zeta_1 \right. \\
&\qquad \left. - \oint_{\Gamma_2} \varphi_2(\zeta_2) R(\zeta_2) \oint_{\Gamma_1} \frac{\varphi_1(\zeta_1)}{\zeta_1 - \zeta_2}\, d\zeta_1\, d\zeta_2 \right]
\end{aligned}
$$

Since ζ_1 is inside Γ_2 and ζ_2 is outside Γ_1, the CIF gives

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{\varphi_1(\zeta_1)}{\zeta_1-\zeta_2}\,d\zeta_1, \qquad \text{and} \\ \varphi_2(\zeta_1) &= \frac{1}{2\pi i}\oint_{\Gamma_2}\frac{\varphi_2(\zeta_2)}{\zeta_2-\zeta_1}\,d\zeta_2\ . \end{aligned}$$

Therefore,

$$\varphi_1(L)\circ\varphi_2(L) = -\frac{1}{2\pi i}\oint_{\Gamma_1}\varphi_1(\zeta_1)\varphi_2(\zeta_1)R(\zeta_1)\,d\zeta_1 = (\varphi_1\varphi_2)(L)\ .$$

Remarks

(1) Since $(\varphi_1\varphi_2)(\zeta) = (\varphi_2\varphi_1)(\zeta)$, (b) implies that $\varphi_1(L)$ and $\varphi_2(L)$ always commute.

(2) Suppose $L \in \mathcal{L}(V)$ is invertible and $\varphi(\zeta) = \frac{1}{\zeta}$. Since $\Sigma(L) \subset \mathbb{C} \setminus \{0\}$ and φ is holomorphic on $\mathbb{C} \setminus \{0\}$, $\varphi(L)$ is defined. Since $\zeta \cdot \frac{1}{\zeta} = \frac{1}{\zeta} \cdot \zeta = 1$, $L\varphi(L) = \varphi(L)L = I$. Thus $\varphi(L) = L^{-1}$, as expected.

(3) Similarly, one can show that if

$$\varphi(\zeta) = \frac{p(\zeta)}{q(\zeta)}$$

is a rational function, i.e.p, q are polynomials, and

$$\Sigma(L) \subset \{\zeta : q(\zeta) \neq 0\},$$

then

$$\varphi(L) = p(L)q(L)^{-1},$$

as expected.

The Spectral Mapping Theorem

Suppose $L \in \mathcal{L}(V)$ and φ is holomorphic in a neighborhood of $\Sigma(L)$ (so $\varphi(L)$ is well-defined). Then

$$\Sigma(\varphi(L)) = \varphi(\Sigma(L))$$

including multiplicities, i.e., if $\mu_1, \ldots, \mu_n$ are the eigenvalues of L counting multiplicities, then $\varphi(\mu_1), \ldots, \varphi(\mu_n)$ are the eigenvalues of $\varphi(L)$ counting multiplicities.

Proof

Let Ω be the domain on which φ is holomorphic and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of L, with algebraic multiplicities $m_1, \ldots, m_k$, respectively. Let Γ be the union of k simple closed curves Γ_j, where each Γ_j is a circle around λ_j and such that the disc Δ_j associated with Γ_j is contained in Ω. By the residue theorem,

$$\varphi(L) = -\frac{1}{2\pi i} \oint_\Gamma \varphi(\zeta) R(\zeta) d\zeta = -\sum_{i=1}^{k} \mathcal{R}es_{\zeta=\lambda_1}[\varphi(\zeta)R(\zeta)].$$

By the partial fractions decomposition of the resolvent,

$$-R(\zeta) = \sum_{i=1}^{k} \left(\frac{P_i}{\zeta - \lambda_i} + \sum_{\ell=1}^{m_i-1} (\zeta - \lambda_i)^{-(\ell+1)} N_i^\ell \right).$$

It follows that

$$\begin{aligned} -\mathcal{R}es_{\zeta=\lambda_i}\varphi(\zeta)R(\zeta) &= \varphi(\lambda_i)P_i + \sum_{\ell=1}^{m_i-1} \mathcal{R}es_{\zeta=\lambda_i}[\varphi(\zeta)(\zeta-\lambda_i)^{-(\ell+1)}]N_i^\ell \\ &= \varphi(\lambda_i)P_i + \sum_{\ell=1}^{m_i-1} \frac{1}{\ell!}\varphi^{(\ell)}(\lambda_i)N_i^\ell \ . \end{aligned}$$

Proof

Thus

$$\varphi(L) = \sum_{i=1}^{k} [\varphi(\lambda_i) P_i + \sum_{\ell=1}^{m_i - 1} \frac{1}{\ell!} \varphi^{(\ell)}(\lambda_i) N_i^{\ell}]$$

By the uniqueness of the spectral decomposition of an operator, this must be the explicit formula for the spectral decomposition of $\varphi(L)$!

Spectral Properties of Composition

Proposition. Let $L \in \mathcal{L}(V)$.
Suppose φ_1 is holomorphic in a neighborhood of $\Sigma(L)$, and φ_2 is holomorphic in a neighborhood of

$$\Sigma(\varphi_1(L)) = \varphi_1(\Sigma(L)).$$

So $\varphi_2 \circ \varphi_1$ is holomorphic in a neighborhood of $\Sigma(L)$.
Then

$$(\varphi_2 \circ \varphi_1)(L) = \varphi_2(\varphi_1(L)).$$

proof

Let Δ_2 a the union of discs containing $\Sigma(\varphi_1(L))$ with φ_2 holomorphic on Δ_2, and let $\Gamma_2 = \partial\Delta_2$. Then

$$\varphi_2(\varphi_1(L)) = \frac{1}{2\pi i}\oint_{\Gamma_2} \varphi_2(\zeta_2)(\zeta_2 - \varphi_1(L))^{-1} d\zeta_2.$$

For each fixed $\zeta_2 \in \Gamma_2$, consider the function

$$f(\zeta_1) = (\zeta_2 - \varphi_1(\zeta_1))^{-1}\ .$$

Let Δ_1 be a union of discs containing $\Sigma(L)$ and chosen so small that φ_1 holomorphic on Δ_1 and $\varphi_1(\Delta_1) \subset \mathrm{int}(\Delta_2)$. Set $\Gamma_1 = \partial\Delta_1$. Then at each point of Γ_2, the function f is holomorphic on an open set containing Δ_1. Therefore, by the operator version of the Cauchy integral formula,

$$f(L) = (\zeta_2 - \varphi_1(L))^{-1} = -\frac{1}{2\pi i}\oint_{\Gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1\ .$$

Plugging the expression

$$(\zeta_2 - \varphi_1(L))^{-1} = -\frac{1}{2\pi i} \oint_{\Gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1$$

into

$$\varphi_2(\varphi_1(L)) = \frac{1}{2\pi i} \oint_{\Gamma_2} \varphi_2(\zeta_2)(\zeta_2 - \varphi_1(L))^{-1} d\zeta_2$$

gives

$$\begin{aligned}
\varphi_2(\varphi_1(L)) &= -\frac{1}{(2\pi i)^2} \oint_{\Gamma_2} \varphi_2(\zeta_2) \oint_{\Gamma_1} (\zeta_2 - \varphi_1(\zeta_1))^{-1} R(\zeta_1) d\zeta_1 d\zeta_2 \\
&= -\frac{1}{(2\pi i)^2} \oint_{\Gamma_1} R(\zeta_1) \oint_{\Gamma_2} \frac{\varphi_2(\zeta_2)}{\zeta_2 - \varphi_1(\zeta_1)} d\zeta_2 d\zeta_1 \\
&= -\frac{1}{2\pi i} \oint_{\Gamma_1} R(\zeta_1) \varphi_2(\varphi_1(\zeta_1)) d\zeta_1 \\
&= (\varphi_2 \circ \varphi_1)(L).
\end{aligned}$$

□

Logarithms of Invertible Matrices

As an application of the theory given above, we consider logarithms of invertible matrices.

Let $L \in \mathcal{L}(V)$ be invertible. We could define the logarithm using power series. That is, one could define

$$\log(I + L) = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{L^\ell}{\ell}.$$

But this series only converges absolutely in norm for a restricted class of L, namely $\{A : \rho(A) < 1\}$.

Let us now take an operator approach. Choose a branch of $\log \zeta$ holomorphic in a neighborhood of $\sigma(L)$. Next choose an appropriate region Ω in which $\log \zeta$ is defined. In this context, by *region*, we mean that Ω is open and $\Gamma = \partial\Omega$ is a simple closed curve.

Form

$$\log L = -\frac{1}{2\pi i} \int_\Gamma (\log \zeta) R(\zeta) d\zeta.$$

This definition depends on the particular branch chosen, but since $e^{\log \zeta} = \zeta$ for any such branch, it follows that for any such choice, $e^{\log L} = L$. Hence, **every** invertible matrix is in the range of the exponential!

This is much better than one can do with series.