Linear Analysis
Lecture 18

The Resolvent

Let V be a finite-dimensional vector space and $L \in \mathcal{L}(V)$. If $\zeta \notin \Sigma(L)$, then the operator $L - \zeta I$ is invertible.

Define

$$R(\zeta) = (L - \zeta I)^{-1}$$

(sometimes denoted $R(\zeta, L)$). The function $R : \mathbb{C} \backslash \Sigma(L) \to \mathcal{L}(V)$ is called the **resolvent** of L.

$R(\zeta)$ provides an analytic approach to questions about the spectral theory of L.

The set $\mathbb{C} \backslash \Sigma(L)$ is called the **resolvent set** of L.

Since the inverses of commuting invertible linear transformations also commute,

$$R(\zeta_1) \quad \text{and} \quad R(\zeta_2) \quad \text{commute} \quad \forall \quad \zeta_1, \zeta_2 \in \mathbb{C} \backslash \Sigma(L).$$

Since a linear transformation commutes with its inverse, it also follows that L commutes with $R(\zeta)$ for all values of ζ.

Basic Resolvent Equations

Let $\zeta_1, \zeta_2 \in \mathbb{C}\backslash\Sigma(L)$.

$$\begin{aligned} R(\zeta_1) - R(\zeta_2) &= R(\zeta_1)(L-\zeta_2)R(\zeta_2) - R(\zeta_1)(L-\zeta_1)R(\zeta_2) \\ &= R(\zeta_1)[(L-\zeta_2)-(L-\zeta_1)]R(\zeta_2) \\ &= (\zeta_1-\zeta_2)R(\zeta_1)R(\zeta_2) \end{aligned}$$

$$R(\zeta_1)R(\zeta_2) = (\zeta_1-\zeta_2)^{-1}[R(\zeta_1)-R(\zeta_2)]$$

$$R(\zeta_1) = [I-(\zeta_2-\zeta_1)R(\zeta_1)]R(\zeta_2)$$

$$R(\zeta_2) = [I-(\zeta_2-\zeta_1)R(\zeta_1)]^{-1}R(\zeta_1)$$

$R(\zeta)$ is Holomorphic on $\mathbb{C}\backslash\Sigma(L)$

Let $\zeta, \zeta_0 \in \mathbb{C}\backslash\Sigma(L)$. Apply results for Neumann series:

$$\begin{aligned} R(\zeta) &= [I - (\zeta - \zeta_0)R(\zeta_0)]^{-1} R(\zeta_0) \\ &= \sum_{n=0}^{\infty} (\zeta - \zeta_0)^n R(\zeta_0)^{n+1} \end{aligned}$$

with this series being absolutely convergent if $|\zeta - \zeta_0| < \|R(\zeta_0)\|^{-1}$. This is just the Taylor series expansion of R at $\zeta = \zeta_0$. Hence R is holomorphic on $\mathbb{C}\backslash\Sigma(L)$ with

$$R^{(n)}(\zeta) = n! R(\zeta)^{n+1} \quad n = 1, 2, \ldots\ .$$

In addition, for $|\zeta|$ large, $|\zeta|^{-1}\|L\| < 1$,

$$R(\zeta) = -\zeta^{-1}(1 - \zeta^{-1}L)^{-1} = -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n$$

which is absolutely convergent if $|\zeta| > \|L\|$.

Thus, $R(\zeta)$ is holomorphic at ∞, and $R(\zeta) \to 0$ as $\zeta \to \infty$.

Cauchy Integral Formulas and Laurent Series

Cauchy Integral Formulas: If f is analytic inside and on a simple closed curve C and z is any point inside C, then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Moreover, the n^{th} derivative of f at z is given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^n} d\zeta \;, \qquad n = 0, 1, 2, \ldots \;.$$

Laurent Series Expansions: If f is analytic inside and on the boundary of the annular shaped region R bounded by two concentric circles C_1 and C_2 with center at z_0 and respective radii r_1 and r_2 $(r_1 > r_2)$, then for all z in R,

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n = P + A$$

where

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \quad n = 0, 1, 2, \ldots$$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{(\zeta - a)^{-n+1}} d\zeta, \quad n = 1, 2, \ldots$$

$P =$ **principal part** $\qquad A =$ **analytic part**

Singularities

Poles: If $f(z)$ has Laurent expansion in which the principle part has only finitely many terms given by

$$\frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-n}}{(z-z_0)^n}$$

where $a_{-n} \neq 0$, then z_0 is a pole of order n for f. If $n = 1$, then it is a simple pole.

Essential Singularities: If $f(z)$ has Laurent expansion in which the principle part has infintely many terms, then z_0 is an essential singularity for f.
A function is **entire** if it is analytic on all of $\mathbb{C}$. If f is analytic on a region Ω except for finitely many poles, then f is said to be **meromorphic** on Ω.
The coefficient a_{-1} is called the **residue** of f.

The Residue Theorem

Let f be single-valued and analytic inside and on a simple closed curve C except at the

$$\text{singularities} \qquad z_1, \ldots, z_n$$

inside C having

$$\text{residues} \qquad a_{-1}^1, a_{-1}^2, \ldots, a_{-1}^n \ .$$

Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} a_{-1}^k.$$

The Laurent Series of the Resolvent

The resolvent is meromorphic with poles at each eigenvalue. For the sake of simplicity we assume that $0 \in \Sigma(L)$ and we compute the Laurent series at the origin:

$$R(\zeta) = \sum_{n=-\infty}^{\infty} \zeta^n A_n,$$

where

$$A_n = \frac{1}{2\pi i} \oint_\Gamma \zeta^{-(n+1)} R(\zeta) d\zeta \qquad \forall\ n.$$

Let $\tilde{\Gamma}$ be a contour around the origin slightly larger than Γ, then

$$\begin{aligned} A_n A_m &= \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} \omega^{-(m+1)} R(\omega) R(\zeta) d\omega d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} [R(\omega) - R(\zeta)] d\omega d\zeta \ . \end{aligned}$$

The Laurent Series of the Resolvent: Proof

For $|\zeta| < |\omega|$, or $\left|\frac{\zeta}{\omega}\right| < 1$,

$$\zeta^{-(n+1)}(\omega-\zeta)^{-1} = \zeta^{-(n+1)}\omega^{-1}\left(1-\frac{\zeta}{\omega}\right)^{-1} = \zeta^{-(n+1)}\omega^{-1}\sum_{k=0}^{\infty}\left(\frac{\zeta}{\omega}\right)^{k}$$

$$= \sum_{k=0}^{\infty}\zeta^{k-(n+1)}\omega^{-(k+1)} = \sum_{j=-n}^{\infty}\zeta^{j-1}\omega^{-(j+n+1)},$$

so for $n \geq 0$ the residue in ζ occurs for $j = 0$ which gives the residue $a_{-1} = \omega^{-(n+1)}$. By the Residue Theorem and geometric series,

$$\begin{aligned} \eta_n\omega^{-(n+1)} &= \frac{1}{2\pi i}\oint_{\Gamma}\zeta^{-(n+1)}(\omega-\zeta)^{-1}\,d\zeta \\ &= -(1-\eta_m)\zeta^{-(m+1)}\frac{1}{2\pi i}\oint_{\tilde{\Gamma}}\omega^{-(m+1)}(\omega-\zeta)^{-1}\,d\omega\ , \end{aligned}$$

with

$$\eta_n = \begin{cases} 1, & \text{for} \quad n \geq 0 \\ 0, & \text{else.} \end{cases}$$

Therefore,

$$\begin{aligned}
A_n A_m &= \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} [R(\omega) - R(\zeta)] \, d\omega \, d\zeta \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} R(\omega) \, d\omega \, d\zeta \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} \omega^{-(m+1)} (\omega - \zeta)^{-1} R(\zeta) \, d\omega \, d\zeta \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \omega^{-(m+1)} R(\omega) \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} (\omega - \zeta)^{-1} \, d\zeta \, d\omega \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \oint_{\tilde{\Gamma}} \zeta^{-(n+1)} R(\zeta) \oint_\Gamma \omega^{-(m+1)} (\omega - \zeta)^{-1} \, d\omega \, d\zeta \\
&= \eta_n \frac{1}{2\pi i} \oint_\Gamma \omega^{-((m+n+1)+1)} R(\omega) \, d\omega \\
&\quad + (\eta_m - 1) \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} \zeta^{-((m+n+1)+1)} R(\zeta) \, d\zeta \\
&= (\eta_m + \eta_n - 1) A_{m+n+1} \ .
\end{aligned}$$

The Laurent Series of the Resolvent

n	m	n+m+1	$A_nA_m = (\eta_n + \eta_m - 1)A_{n+m+1}$	
-1	-1	-1	$A_{-1}^2 = -A_{-1}$	$-A_{-1} =: P$ a projection
			$N := -A_2$	
-2	-2	-3	$A_{-2}^2 = -A_{-3}$	$-A_{-3} = N^2$
-2	-3	-4	$A_{-2}A_{-3} = -A_{-4}$	$-A_{-4} = N^3$
-2	-4	-5	$A_{-2}A_{-4} = -A_{-5}$	$-A_{-5} = N^4$
			$\vdots$	$-A_{-k} = N^{k-1}$

The Laurent Series of the Resolvent

n	m	n+m+1	$A_n A_m = (\eta_n + \eta_m - 1) A_{n+m+1}$	
			$S := A_0$	
0	0	1	$A_0^2 = A_1$	$A_1 = S^2$
0	1	2	$A_0 A_1 = A_2$	$A_2 = S^3$
0	2	3	$A_0 A_2 = A_3$	$A_3 = S^4$
			$\vdots$	$A_n = S^{n+1} \quad n \geq 0$

The Laurent Series of the Resolvent

Therefore, for $\lambda_j \in \Sigma(T) \quad \exists\, P_j, N_j, S_j$ such that the Laurent series expansion for $R(\zeta)$ near λ_j is

$$\begin{aligned} R(\zeta) &= \left[-(\zeta - \lambda_j)^{-1} P_j - \sum_{n=1}^{\infty} (\zeta - \lambda_j)^{-(n+1)} N_j^n \right] \\ &\quad + \left[\sum_{n=0}^{\infty} (\zeta - \lambda_j)^n S_j^{(n+1)} \right] \\ &= C_j(\zeta) + S_j(\zeta)\ . \end{aligned}$$

The Laurent Series of the Resolvent

Moreover,

n	m	n+m+1	$A_n A_m = (\xi_n + \xi_m - 1) A_{n+m+1}$
-1	-2	-2	$N_j = -A_{-2} = A_{-1} A_{-2} = P_j N_j$
-2	-1	-2	$N_j = -A_{-2} = A_{-2} A_{-1} = N_j P_j$
-1	0	0	$0 = 0 \cdot A_0 = A_{-1} A_0 = -P_j S_j$
0	-1	0	$0 = 0 \cdot A_0 = A_0 A_{-2} = -S_j P_j$

Hence, for each $\lambda_j \in \Sigma(A)$, the decomposition $R(\zeta) = C_j(\zeta) + S_j(\zeta)$ is a direct sum decomposition of R compatible with $V = M_j \oplus M_j'$, where

$$M_j = P_j V \quad \text{and} \quad M_j' = (I - P_j) V \ .$$

The Laurent Series of the Resolvent

The principal part of the Laurent expansion for $R(\zeta)$ at $\lambda_j \in \Sigma(L)$ acts on the subspace M_j. In particular, $R(\zeta)$ has an isolated singularity at $\zeta = \lambda_j$ but is otherwise convergent. Thus,

$$\sum_{h=1}^{\infty} (\zeta - \lambda_j)^{-(n+1)} N_j^n$$

is absolutely convergent on $\mathbb{C} \backslash \{\lambda_j\}$. Setting $\zeta = \lambda_j + \xi$ for $\xi \neq 0$, we obtain

$$\sum_{n=1}^{\infty} \xi^{-(n+1)} N_j^n < \infty.$$

Therefore, $\rho(N) \leq |\xi|$ for all $\xi \in \mathbb{C}$.
Consequently, $\rho(N) = 0$ and so N_j is nilpotent with

$$\operatorname{rank}(N_j) < \operatorname{rank}(P_j)$$

where

$$\operatorname{rank}(P_j) = \dim(M_j) = m_j.$$

Hence, λ_j is a pole of $R(\zeta)$ of order less than or equal to $\operatorname{rank}(P_j)$ since the principal part the Laurent series at λ_j is finite.
Therefore, $R(\zeta)$ is meromorphic!

How are the P_j's and N_j's related?

Claim: If $\Sigma(L) = \{\lambda_1, \ldots, \lambda_s\}$, then

$$P_j P_k = \delta_{jk} P_j \tag{1}$$

$$\sum_{j=1}^{s} P_j = I \tag{2}$$

$$P_j L = L P_j \; . \tag{3}$$

Proof of (1)

(1) Show $P_j P_k = \delta_{jk} P_j$.

$$P_j P_k = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_j} \int_{\Gamma_k} R(\zeta) R(w) dw d\zeta,$$

where the regions defined by Γ_j and Γ_k do not overlap.

$$\begin{aligned} P_j P_k &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_j} \int_{\Gamma_k} (w-\zeta)^{-1} [R(w) - R(\zeta)] dw d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^2 \left[\int_{\Gamma_k} \left[\int_{\Gamma_j} (w-\zeta)^{-1} R(w) d\zeta \right] dw \right. \\ &\qquad \left. - \int_{\Gamma_j} \left[\int_{\Gamma_k} (w-\zeta)^{-1} R(\zeta) dw \right] d\zeta \right] \\ &= 0 \qquad j \neq k. \end{aligned}$$

Proof of (2)

(2) Show $\sum_{j=1}^{s} P_j = I$.

Let Γ be a simple closed curve containing all the singularities of $R(\zeta)$. Then

$$\frac{1}{2\pi i}\int_\Gamma R(\zeta)ds = \text{ sum of the residues } = -\sum_{j=1}^{s} P_j.$$

Also, from the expansion of $R(\zeta)$ at ∞, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma R(\zeta)ds &= \frac{1}{2\pi i}\int_\Gamma -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n ds \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} -L^n \left(\int_\Gamma \zeta^{-(n+1)} d\zeta\right) \\ &= -I. \end{aligned}$$

Proof of (3)

(3) Show $P_j L = L P_j$.

L commutes with $R(\zeta)$ so that L commutes with

$$P_j = \frac{1}{2\pi i} \int_{\Gamma_j} R(\zeta) d\zeta.$$

Properties of the Resolvent

Let V be a finite-dimensional vector space and $L \in \mathcal{L}(V)$. If $\zeta \notin \sigma(L)$, then the operator $L - \zeta I$ is invertible. We have defined the resolvent of L as

$$R(\zeta) = (L - \zeta I)^{-1} \ .$$

We have shown that for each $\lambda_j \in \Sigma(L) \quad \exists\, P_j, N_j, S_j$ such that

$$\begin{aligned} R(\zeta) &= \left[-(\zeta - \lambda_j)^{-1} P_j - \sum_{n=1}^{m_j} (\zeta - \lambda_j)^{-(n+1)} N_j^n \right] \\ &\quad + \left[\sum_{n=0}^{\infty} (\zeta - \lambda_j)^n S_j^{(n+1)}\right] \\ &= C_j(\zeta) + S_j(\zeta) \ , \end{aligned}$$

where $m_j = \operatorname{rank}(P_j)$.

$$N_j P_j = N_j = P_j N_j, \qquad P_j S_j = 0 = S_j P_j, \qquad P_j P_k = \delta_{jk} P_j$$

and

$$P_j L = L P_j \ .$$

Partial Fractions Decomposition of $R(\zeta)$

Let $C_j(\zeta)$ be the principal part of $R(\zeta)$ at each of its poles $\Sigma(L) = \{\lambda_1,\ \lambda_2,\ \dots,\ \lambda_s\}$. Then the function

$$F(\zeta) = R(\zeta) - \sum_{j=1}^{s} C_j(\zeta)$$

has an analytic extension to all of $\mathbb{C}$ since the poles λ_j are removable. Moreover,

$$\lim_{\zeta\to\infty} F(\zeta) = 0$$

since

$$\lim_{\zeta\to\infty} C_j(\zeta) = 0 \quad \text{and} \quad \lim_{\zeta\to\infty} R(\zeta) = 0 \quad \text{we have} \quad R(\zeta) = -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n \,.$$

Therefore, $F(\zeta)$ is a bounded entire function. By Liouville's Theorem $F \equiv 0$. Hence,

$$R(\zeta) \;=\; \sum_{j=1}^{s} C_j(\zeta) = -\sum_{j=1}^{s} \left[(\zeta-\lambda_j)^{-1} P_j + \sum_{n=1}^{m_j-1} (\zeta-\lambda_j)^{-(n+1)} N_j^n \right] \,.$$

Spectral Representation Theorem

Since $X = M_1 \oplus \cdots \oplus M_s$, we can write L as the sum of the operators $P_j L P_j$ $j = 1, \ldots, s$. Note that for each $j = 1, \ldots, s$

$$P_j L P_j = L P_j = \lambda_j P_j + N_j.$$

Therefore, $L = S + N$ where $S = \sum_{j=1}^{s} \lambda_j P_j$ and $N = \sum_{j=1}^{s} N_j$, with

$$\begin{aligned} I &= \sum_{j=1}^{s} P_j \\ P_k P_j &= P_j P_k = \delta_{jk} P_j \\ N_j P_k &= P_k N_j = \delta_{kj} N_j \\ N_j P_k &= P_k N_j = 0 \quad i \neq j \ . \end{aligned}$$

This representation is unique in the sense that any other such representation $L = S' + N'$ has $S = S'$ and $N = N'$.