Linear Analysis
Lecture 14

The Projection Theorem for Convex Sets

Let X be a Hilbert space with $C \subset X$ closed and convex.
Then there is a unique $y^0 \in C$ such that

$$\|x - y^0\| \leq \|x - y\| \qquad \forall\ y \in C. \tag{$\mathcal{P}$}$$

Furthermore, y_0 satisfies $\mathcal{P}$ if and only if

$$\text{Re}(\langle x - y^0, y - y^0 \rangle) \leq 0 \qquad \forall\ y \subset C.$$

Proof of The Projection Theorem for Convex Sets

Let $\{y^i\} \subset C$ be such that

$$\|x - y^i\| \to \inf\{\|x - y\| : y \in C\} =: \delta.$$

By the parallelogram law

$$\|y^m - y^n\|^2 = 2\|x - y^m\|^2 + 2\|x - y^n\|^2 - 4\left\|x - \frac{y^n + y^m}{2}\right\|^2.$$

By convexity, $2^{-1}(y^n + y^m) \in C$, so $\|x - 2^{-1}(y^m + y^n)\| \geq \delta$. Therefore,

$$\|y^m - y^n\|^2 \leq 2\|y^m - x\|^2 + 2\|y^n - x\|^2 - 4\delta^2 \to 0.$$

Consequently, $\{y^n\}$ is Cauchy and so has a limit y^0 with $\|x - y^0\| = \delta$.

To see that y^0 is unique consider the sequence

$$y^{2n+1} = y^a \quad \text{and} \quad y^{2n} = y^b \quad n = 0, 1, \ldots$$

where $y^a, y^b \in C$ with $\|h^a - x\| = \|y^b - x\| = \delta$. If we apply the parallelogram law to this sequence, we find that it is Cauchy as above. Hence, $y^a = y^b$.

Proof of The Projection Theorem for Convex Sets

We now show that y^0 is the unique vector satisfying

$$\text{Re}(\langle x - y^0, y - y^0 \rangle) \leq 0 \text{ for all } y \in C.$$

Suppose to the contrary that there is a vector y^1 such that

$$\text{Re}(\langle x - y^0, y^1 - y^0 \rangle) = \epsilon > 0.$$

Consider the vectors

$$y^\alpha = \alpha y^1 + (1 - \alpha) y^0 \in C \quad \text{for} \quad \alpha \in [0, 1].$$

Note that the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(\alpha) = \|x - y^\alpha\|^2 =$$
$$(1 - \alpha)^2 \|x - y^0\|^2 \quad + 2\alpha(1 - \alpha)\text{Re}(\langle x - y^0, x - y^1 \rangle) + \alpha^2 \|x - y^1\|^2$$

is differentiable with

$$\begin{aligned} \varphi'(0) &= -2\|x - y^0\|^2 + 2\text{Re}(\langle x - y^0, x - y^1 \rangle) \\ &= -2\text{Re}(\langle x - y^0, x - y^0 \rangle + \langle x - y^0, y^1 - x \rangle) \\ &= -2\text{Re}\langle x - y^0, y^1 - y^0 \rangle = -2\epsilon < 0. \end{aligned}$$

Hence, $\|x - y^\alpha\| < \|x - y^0\|$ for all $\alpha > 0$ sufficiently small. This contradiction implies that y^1 does not exist.

Proof of The Projection Theorem for Convex Sets

Conversely, suppose that $y^0 \in C$ is such that

$$\text{Re}(\langle x - y^0, y - y^0 \rangle) \le 0 \quad \forall\ y \in C.$$

Then for any $y \in C$ with $y \neq y^0$, we have

$$\begin{aligned} \|x - y\|^2 &= \|(x - y^0) + (y^0 - y)\|^2 \\ &= \|x - y^0\|^2 + 2\text{Re}(\langle x - y^0, y^0 - y \rangle) + \|y^0 - y\|^2 \\ &> \|x - y^0\|^2. \end{aligned}$$

□

Affine Sets

A subset W of a vector space V is said to *affine* if

$$(1-\lambda)u + \lambda v \in W \quad \forall\ u, v \in W\ \lambda \in \mathbb{F}\ .$$

Fact: A subset W of V is affine iff there is a subspace S such that

$$W = w + S = \{w + x\ :\ x \in S\} \quad \forall\ w \in W\ .$$

Proof

$(\Leftarrow)$ First consider sets of the form $W = w + S$ for $w \in V$ and S a subspace. For any $u, v \in W$ and $\lambda \in \mathbb{F}$,

$$\exists\ x, y \in S \text{ with } u = w + x,\ v = w + y$$

and so

$$\begin{aligned} (1-\lambda)u + \lambda v &= (1-\lambda)(w+x) + \lambda(w+y) \\ &= w + (1-\lambda)x + \lambda y \in w + S\ . \end{aligned}$$

Therefore, W is affine.

Also, for any $\bar{w} \in W$ with $\bar{w} = w + x$, we have

$$\bar{w} + S = w + x + S = w + S,$$

since S is a subspace.

Affine Sets

($\Rightarrow$) Let $\bar{w} \in W$ and set $S = \{w - \bar{w} \, : \, w \in W\}$. We claim S is a subspace.

Clearly $0 \in S$. Also, given any $\alpha \in \mathbb{F}$ and $x \in S$, there is a $u \in W$ such that $x = u - \bar{w}$ and so

$$\alpha x = \alpha(u - \bar{w}) = \alpha u + (1-\alpha)\bar{w} - \bar{w} \in W - \bar{w} = S \; ,$$

so $\alpha x \in S$ for all $\alpha \in \mathbb{F}$ whenever $x \in S$. Moreover, for any $x, y \in S$,

$$\exists \; u, v \in W \text{ with } x = u - \bar{w} \text{ and } y = v - \bar{w}$$

so

$$\frac{1}{2}(x+y) = \frac{1}{2}u + \frac{1}{2}v - \bar{w} \in W - \bar{w} = S,$$

but then $x + y \in S$, so S is a subspace. Therefore, $W = \bar{w} + S$ and by what we have already shown

$$S = W - w \qquad \forall \; w \in W.$$

$\square$

Prime Example of an Affine Set

Let $A \in \mathcal{B}(U, V)$ and $b \in V$, where U and V are vector spaces over $\mathbb{F}$. Then

$$W = \{x \in V \,:\, Ax = b\}$$

is an affine subset of W, with

$$W = \bar{w} + \mathcal{N}(A),$$

where $\bar{w}$ is any particular solution to the equation $Ax = b$.

Projections onto Affine Sets

Let $W = \hat{w} + S$ be an affine subset of the Euclidean space V. Then
(i) For every $v \in V$ there exists a unique solution $\bar{w}$ to the problem

$$(\diamondsuit) \qquad \underset{w \in W}{\text{minimize}} \ \|v - w\|^2 \ .$$

(ii) $\bar{w}$ solves $(\diamondsuit)$ iff $v - \bar{w} \in S^\perp$.
(iii) $\bar{w} = Pv + (I - P)\hat{w}$ solves $(\diamondsuit)$ where P is the orthogonal projection onto S.
Proof: $(\diamondsuit)$ is equivalent to the problem

$$\underset{w \in \hat{w}+S}{\text{minimize}} \ \|v - w\|^2 \quad = \quad \underset{x \in S}{\text{minimize}} \ \|(v - \hat{w}) - x\|^2 \ .$$

By the Projection Theorem, there is a unique $\hat{x} \in S$ solving this problem, in which case $\bar{w} = \hat{w} + \hat{x}$ is the unique solution to $(\diamondsuit)$. Moreover, again by the Projection Theorem, $\hat{x}$ is the unique element of S such that

$$v - \bar{w} = (v - \hat{w}) - \bar{x} \in S^\perp \ .$$

To see (iii) note that $\hat{x} = P(v - \hat{w})$, so

$$\bar{w} = \hat{w} + \hat{x} = \hat{w} + P(v - \hat{w}) = Pv + (I - P)\hat{w} \ . \qquad \square$$