Linear Analysis
Lecture 1

Vector Spaces

$\mathbb{F}$ $\sim$ The field of scalars, always $\mathbb{R}$ or $\mathbb{C}$.

V $\sim$ A vector space, that is a nonempty set on which is defined the operations of

addition (for $v,\ w \in V$, $v + w \in V$) and

scalar multiplication (for $\alpha \in \mathbb{F}$ and $v \in V$, $\alpha v \in V$).

Vector Space Axioms

The operations on a vector space satisfy the following axioms:
for every $x, y, z \in V$ and $\lambda, \mu \in \mathbb{F}$

1. $(x + y) + z = x + (y + z)$,
2. $x + y = y + x$,
3. $\exists\, 0 \in V$ such that $x + 0 = x$,
4. $\forall\, x \in V$ there exists $z \in V$ such that $x + z = 0$ (written $z = -x$),
5. $\lambda x = x\lambda$,
6. $\lambda(x + y) = \lambda x + \lambda y$,
7. $(\lambda + \mu)x = \lambda x + \mu x$,
8. $\lambda(\mu x) = (\lambda\mu)x$, and
9. $1x = x$.

Examples of Vector Spaces

(0) $\{0\}$

(1) $\mathbb{F}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : \text{ each } x_j \in \mathbb{F} \right\}, n \geq 1$

(2) $\mathbb{F}^\infty = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} : \text{each } x_j \in \mathbb{F} \right\}$

- $\ell^1(\mathbb{F}) \subset \mathbb{F}^\infty$, where $\ell^1(\mathbb{F}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} : \sum_{j=1}^{\infty} |x_j| < \infty \right\}$
- $\ell^\infty(\mathbb{F}) \subset \mathbb{F}^\infty$, where $\ell^\infty(\mathbb{F}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} : \sup_j |x_j| < \infty \right\}$
- $\ell^1(\mathbb{F})$ and $\ell^\infty(\mathbb{F})$ are clearly subspaces of $\mathbb{F}^\infty$.
- Let $0 < p < \infty$, and define $\ell^p(\mathbb{F}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} : \sum_{j=1}^{\infty} |x_j|^p < \infty \right\}$.

Since

$$\begin{aligned} |x+y|^p &\leq (|x|+|y|)^p \leq (2\max(|x|,|y|))^p \\ &= 2^p \max(|x|^p, |y|^p) \leq 2^p(|x|^p + |y|^p), \end{aligned}$$

we have $\ell^p(\mathbb{F}) \subsetneq \mathbb{F}^\infty$. Show that $\ell^p(\mathbb{F}) \subsetneq \ell^q(\mathbb{F})$ if $0 < p < q \leq \infty$.

Examples of Vector Spaces

(3) $X \neq \emptyset$, $\mathcal{F}(X) = \{f : f : X \mapsto \mathbb{F}\}$ all functions from X to $\mathbb{F}$
For $f_1, f_2 \in \mathcal{F}$ and $\in \mathbb{F}$ define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{and} \quad (f)(x) = f(x).$$

(4) X a metric space.

- $C(X, \mathbb{F})$ the set of all continuous $\mathbb{F}$-valued functions on X. $C(X, \mathbb{F})$ is a subspace of $\mathcal{F}(X)$.
- $C_b(X, \mathbb{F}) \subset C(X, \mathbb{F})$ the subspace of all bounded continuous functions $f : X \to \mathbb{F}$.
- $C^k(X, F) \subset C(X, \mathbb{F})$ the subspace of all k times continuously differentiable functions $f : X \to \mathbb{F}$. When $X = \mathbb{F}$ we simply write

  $$C(\mathbb{F}),\ C_b(\mathbb{F}),\ \text{and}\ C^k(\mathbb{F}),$$

  respectively.

(5) Define $\mathcal{P}(\mathbb{F}) \subset C(\mathbb{R}, \mathbb{F})$ to be the space of all $\mathbb{F}$-valued polynomials on $\mathbb{R}$:

$$\mathcal{P}(\mathbb{F}) = \{a_0 + a_1 x + \cdots + a_m x^m : m \geq 0,\ \text{each}\ a_j \in \mathbb{F}\}.$$

Each $p \in \mathcal{P}(\mathbb{F})$ is viewed as a function $p : \mathbb{R} \to \mathbb{F}$ (or $p : \mathbb{F} \to \mathbb{F}$) given by $p(x) = a_0 + a_1 x + \cdots + a_m x^m$.

Examples of Vector Spaces

(6) Define $\mathcal{P}_n(\mathbb{F}) \subset \mathcal{P}(\mathbb{F})$ to be the subspace of all polynomials of degree $\leq n$.

(7) $V = \{u \in C^2(\mathbb{F}) : u'' + u = 0\}$. Then V is a subspace of $C^2(\mathbb{F})$. For $\mathbb{F} = \mathbb{C}$,

$$\begin{aligned} V &= \{a_1 \cos x + a_2 \sin x : a_1, a_2 \in \mathbb{C}\} \\ &= \{b_1 e^{ix} + b_2 e^{-ix} : b_1, b_2 \in \mathbb{C}\} . \end{aligned}$$

More generally, if

$$\begin{aligned} L(u) &= [D^{(m)} + a_{m-1} D^{(m-1)} + \cdots + a_1 D + a_0 I](u) \\ &= u^{(m)} + a_{m-1} u^{(m-1)} + \cdots + a_1 u' + a_0 u \end{aligned}$$

is an m^{th} order linear constant-coefficient differential operator, then

$$V = \{u \in C^m(\mathbb{F}) : L(u) = 0\}$$

is a vector space.
V can be explicitly described as the set of all linear combinations of certain functions of the form $x^j e^{rx}$, where $j \geq 0$ and r is a root of the characteristic polynomial

$$r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0 = 0.$$

Span, Linear Independence, Basis

V a vector space with $S \subset V$. A *Linear combination* of elements of S is any element of V of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_k v_k, \tag{1}$$

where $k \in \mathbb{N}$, $\{v_1, \ldots, v_k\} \subset S$, and $\{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{F}$.

The *linear span* of S, $\mathrm{Span}(S)$, is the set of all linear combinations of finite subsets of S. $\mathrm{Span}(S)$ is a subspace of V.

$S := \{v_1, \ldots, v_k\}$ is said to be *linear independent* if

$$\alpha_1 = 0,\ \alpha_2 = 0,\ \ldots,\ a_k = 0$$

whenever

$$0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_k v_k\ .$$

S is called a *basis* of $\mathrm{Span}(S)$ if S is linearly independent.

Facts:

(a) Every vector space has a basis.
In addition, if S is any linearly independent set in V, then there is a basis of V containing S.
The proof in infinite dimensions uses Zorn's lemma and is nonconstructive. Such a basis in infinite dimensions is called a **Hamel basis**. Typically it is impossible to identify a Hamel basis explicitly, and they are of extremely limited use. There are numerous other sorts of "bases" in infinite dimensions which are very useful.

(b) Any two bases of the same vector space V can be put into $1-1$ correspondence.

Define the **dimension of** V (denoted $\dim V$) $\in \{0, 1, 2, \ldots\} \cup \{\infty\}$ as the number of elements in a basis of V.

Standard Bases

The **standard basis** of $\mathbb{F}^n$ is the vectors $e_1, \dots, e_n$, where

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{\text{th}} \text{ entry},$$

so $\dim \mathbb{F}^n = n$. The vectors $e_1, e_2, \dots \in \mathbb{F}^\infty$ are linearly independent. However, $\operatorname{Span}\{e_1, e_2, \dots\}$ is the proper subset of $\mathbb{F}^\infty$ consisting of all vectors with only finitely many nonzero components. We denote this subspace by $\mathbb{F}_0^\infty$. In particular, $\{e_1, e_2, \dots\}$ is not a basis of $\mathbb{F}^\infty$, but is a basis for $\mathbb{F}_0^\infty$.
The monomials $\{x^m : m \in \{0, 1, 2, \dots\}\}$ form a basis of $\mathcal{P}$.

Remark. Any $\mathbb{C}$-vector-space V may be regarded as an $\mathbb{R}$-vector-space by restriction of the scalar multiplication.
If V is finite-dimensional with basis $\{v_1, \dots, v_n\}$ over $\mathbb{C}$, then

$$\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$$

is a basis for V over $\mathbb{R}$. In particular, $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$.

Coordinates

V	–	A finite dim. vector space over $\mathbb{F}$
$\mathcal{B} = \{v_1, \ldots, v_n\}$	–	A basis for V

$v \in V$ can be written uniquely as $v = \sum_{i=1}^{n} x_i v_i$ for some $x_i \in \mathbb{F}$.

Define a map from V into $\mathbb{F}^n$ by $v \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

This is the *coordinate* map w.r.t. the basis $\mathcal{B}$.

The coefficients x_i are called the coordinates of v w.r.t. this basis.

Linear Transformations

Let V and W be vector spaces.

A map $L : V \to W$ is a **linear transformation** if $(\forall\, v_1, v_2 \in V)(\forall\, \alpha_1, \alpha_2 \in \mathbb{F})$

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2).$$

If, in addition, L is bijective (1–1 and onto), then L is called a (vector space) **isomorphism**.

A coordinate mapping on a vector space is an example of a vector space isomorphism.

Every vector space over $\mathbb{F}$ of dim. n is isomorphic to $\mathbb{F}^n$.

Change of Basis

$\mathcal{B}_1 = \{v_1, \ldots, v_n\}$ and $\mathcal{B}_2 = \{w_1, \ldots, w_n\}$ two bases of V.
Let $v \in V$, and suppose $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ are the coordinates of v w.r.t $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively:

$$v = \sum_{i=1}^{n} x_i v_i = \sum_{j=1}^{n} y_j w_j \ .$$

Express each w_j in terms of $\{v_1, \ldots, v_n\}$:

$$w_j = \sum_{i=1}^{n} a_{ij} v_i \quad (a_{ij} \in \mathbb{F}).$$

Set $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{F}^{n\times n}$.

Then

$$\sum_{i=1}^{n} x_i v_i = v = \sum_{j=1}^{n} y_j w_j = \sum_{j=1}^{n} y_j (\sum_{i=1}^{n} a_{ij} v_i) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} y_j \right) v_i,$$

so $x_i = \sum_{j=1}^{n} a_{ij} y_j$, i.e. $x = Ay$.
The matrix A is called the **change of basis** matrix.

Two Views of Matrix-Vector Multiplication

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{F}^{m \times n} \qquad x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{F}^n$$

$$A_{i\cdot} = (a_{i1}, a_{i2}, \ldots, a_{in}) = i^{\text{th}} \text{ row of } A$$

$$A_{\cdot j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = j^{\text{th}} \text{ column of } A$$

$$Ax = \sum_{j=1}^{n} x_j A_{\cdot j} \qquad = \text{lin. combo. of columns of } A$$

$$= \begin{pmatrix} A_{1\cdot} \bullet x \\ A_{2\cdot} \bullet x \\ \vdots \\ A_{m\cdot} \bullet x \end{pmatrix} \qquad = \text{dot product with rows of } A$$

Formal Matrix Notation

Write the basis vectors

$$V = (v_1 \cdots v_n) \quad \text{and} \quad W = (w_1 \cdots w_n)$$

formally as a row of vectors.
Formally, we have

$$\begin{aligned} v_j &= \text{the } j\text{th ``column'' of } V \\ w_j &= \text{the } j\text{th ``column'' of } W \end{aligned}$$

Then

$$w_j = \sum_{i=1}^{n} a_{ij} v_i = (v_1 \cdots v_n) A_{\cdot j}$$

and so

$$(w_1 \cdots w_n) = (v_1 \cdots v_n) A$$

using the usual matrix multiplication rules.
We also have

$$v = (v_1 \cdots v_n) x \quad \text{and} \quad v = (w_1 \cdots w_n) y,$$

so

$$(v_1 \cdots v_n) x = (w_1 \cdots w_n) y = (v_1 \cdots v_n) A y,$$

giving $x = Ay$ as before.

Intersections

$$G \quad -- \quad \text{any index set}$$

$$\{\, W_\gamma : \gamma \in G\} \quad -- \quad \text{a family of subspaces of } V$$

$$W_G \quad = \quad \bigcap_{\gamma \in G} W_\gamma \quad \text{is a subspace of } V.$$

Sums of Subspaces

$$W_1, \ W_2 \quad -- \quad \text{two subspaces of } V$$
$$W_1 + W_2 \quad = \quad \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$$

is a subspace of V, and

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$$

The sum $W_1 + W_2$ is **direct** if

$$W_1 \cap W_2 = \{0\},$$

or, equivalently,

$$\forall\, v \in W_1 + W_2 \ \exists!\, w_1 \in W_1, \ w_2 \in W_2$$
such that
$$v = w_1 + w_2 \ .$$

and we write

$$W_1 + W_2 = W_1 \oplus W_2 \ .$$

More general sums

$$\begin{aligned} W_1, \ldots, W_n \quad &-- \quad \text{subspaces of } V \\ W_1 + \cdots + W_n \quad &= \quad \{w_1 + \cdots + w_n : w_j \in W_j, 1 \leq j \leq n\} \end{aligned}$$

is a subspace of V.

We say the sum is **direct**, written

$$W_1 \oplus \cdots \oplus W_n \ ,$$

if

$$\left[\ w_j \in W_j \ (1 \leq j \leq n) \quad \text{and} \quad \sum_{j=1}^{n} w_j = 0 \ \right]$$

$$\Longrightarrow$$

$$[\ w_j = 0 \ (1 \leq j \leq n) \ ] .$$

Even More general sums

$$\begin{aligned} G \quad &-- \quad \text{an index set} \\ \{W_\gamma : \gamma \in G\} \quad &-- \quad \text{a family of subspaces of } V \\ \sum_{\gamma \in G} W_\gamma \quad &= \quad \operatorname{Span}\left(\bigcup_{\gamma \in G} W_\gamma\right) \end{aligned}$$

is a subspace of V.
The sum is direct, written $\bigoplus_{\gamma \in G} W_\gamma$, if for every finite subset G' of G

$$\left[w_\gamma \in W_\gamma \ (\gamma \in G') \text{ and } \sum_{\gamma \in G'} w_\gamma = 0 \right]$$

$$\Longrightarrow$$

$$[w_\gamma = 0 \ (\gamma \in G')],$$

or, equivalently,

$$\forall\, \beta \in G, \ \ W_\beta \cap \left(\sum_{\gamma \in G, \gamma \neq \beta} W_\gamma \right) = \{0\}.$$

Direct Products

$$\{V_\gamma : \gamma \in G\} \quad -- \quad \text{a family of vector spaces over } \mathbb{F}$$

$$V = \mathop{\times}_{\gamma \in G} V_\gamma$$

$$= \left[\begin{array}{c} \text{All functions } v : G \to \bigcup_{\gamma \in G} V_\gamma \text{ for which} \\ v(\gamma) \in V_\gamma, \quad \forall \gamma \in G. \end{array} \right]$$

Write

$$v_\gamma \quad \text{for} \quad v(\gamma),$$

and

$$v = (v_\gamma)_{\gamma \in G}, \quad \text{or just} \quad v = (v_\gamma).$$

Define

$$v + w = (v_\gamma + w_\gamma) \quad \text{and} \quad \alpha v = (\alpha v_\gamma).$$

Then V is a vector space over $\mathbb{F}$.

Example:

$$G = \mathbb{N} = \{1, 2, \ldots\}, \text{ each } V_n = \mathbb{F}.$$

Then

$$\mathop{\times}_{n \geq 1} V_n = \mathbb{F}^\infty \ .$$

(External) Direct Sums

$\{V_\gamma : \gamma \in G\}$ $--$ a family of vector spaces over $\mathbb{F}$.

$$\bigoplus_{\gamma \in G} V_\gamma = \left[\begin{array}{c} \text{The subspace of } \bigtimes_{\gamma \in G} V_\gamma \\ \text{consisting of those } v \text{ for which} \\ v_\gamma = 0 \text{ except for finitely many } \gamma \in G. \end{array}\right]$$

Example:
For $n = 0, 1, 2, \ldots$ let $V_n = \mathrm{Span}(e_n)$ in $\mathbb{F}_0^\infty$. Then

$$\mathbb{F}_0^\infty = \bigoplus_{n \geq 0} V_n \ .$$

Example:
For $n = 0, 1, 2, \ldots$ let $V_n = \mathrm{Span}(x^n)$ in $\mathcal{P}$. Then

$$\mathcal{P} = \bigoplus_{n \geq 0} V_n \ .$$

(External) Direct Sums

Facts:
(a) If G is a finite index set,

$$\mathop{\times}_{\gamma \in G} V_\gamma \quad \text{and} \quad \bigoplus_{\gamma \in G} V_\gamma$$

are isomorphic.

(b) If each W_γ is a subspace of V and the sum

$$\sum_{\gamma \in G} W_\gamma$$

is direct, then it is naturally isomorphic to the external direct sum

$$\bigoplus_{\gamma \in G} W_\gamma.$$

Quotients

Let W be a subspace of V.

Define on V the equivalence relation

$$[v_1 \sim v_2] \quad \Longleftrightarrow \quad [v_1 - v_2 \in W] \; .$$

Define the **quotient**, denoted V/W, to be the set of equivalence classes given by this equivalence relation.

Let $v + W$ denote the equivalence class of v.

Define a vector space structure on V/W by defining

$$\alpha_1(v_1 + W) + \alpha_2(v_2 + W) = (\alpha_1 v_1 + \alpha_2 v_2) + W \; .$$

The **codimension** of W in V is

$$\operatorname{codim}(W) = \dim(V/W).$$