(1) (a) Let \( f : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C} \) be continuous. Suppose \( x : \mathbb{R} \to \mathbb{C} \) is a solution of the \( n \)-th order equation

\[
(*) \quad x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}),
\]

i.e., for each \( t \in \mathbb{R} \), \( x^{(n)}(t) \) exists and \( x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)) \). Show that \( x \in C^n(\mathbb{R}) \).

(b) Define \( F : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n \) by \( F(t, y) = [y_2, y_3, \ldots, y_n, f(t, y_1, \ldots, y_n)]^T \) (so \( F \) is continuous). Suppose \( y : \mathbb{R} \to \mathbb{C}^n \) is a solution of the first-order system

\[
(**) \quad y' = F(t, y),
\]

i.e., for each \( t \in \mathbb{R} \), \( y'(t) \) exists and \( y'(t) = F(t, y(t)) \). Show that \( y \in C^1(\mathbb{R}) \), and moreover for \( 1 \leq j \leq n \), \( y_j \in C^{n-j+1}(\mathbb{R}) \).

(c) Show that if \( x \in C^n(\mathbb{R}) \) is a solution of \((*)\), then \( y = [x, x', \ldots, x^{(n-1)}]^T \) is a \( C^1 \) solution of \((**)\). Moreover, if \( x \) satisfies the initial conditions

\[
\begin{aligned}
x^{(k)}(t_0) &= x_k^0 \quad (0 \leq k \leq n - 1),
\end{aligned}
\]

then \( y \) satisfies the initial conditions \( y(t_0) = [x_0^0, \ldots, x_0^{n-1}]^T \).

(d) Show that if \( y \) is a \( C^1 \) solution of \((**)\), then \( x = y_1 \) is a \( C^n \) solution of \((*)\).

Moreover, if \( y \) satisfies the initial conditions \( y(t_0) = y_0 \), then \( x \) satisfies the initial conditions \( x^{(k)}(t_0) = (y_0)_k+1 \) \( (0 \leq k \leq n - 1) \).

(e) Show that the first-order system corresponding to the linear \( n \)-th order equation

\[
x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = b(t)
\]

is of the form \( y' = A(t)y + B(t) \) where \( A(t) \in \mathbb{C}^{n \times n} \) and \( B(t) \in \mathbb{C}^n \), and identify \( A(t) \) and \( B(t) \).

(2) Let \( A \in \mathbb{C}^{n \times n} \) have \( \rho(A) < 1 \).

(a) Show by example that \( A \) need not be a contraction with respect to the Euclidean metric.

(b) Show that there is an inner product on \( \mathbb{C}^n \) so that \( A \) is a contraction with respect to the norm induced by the inner product.

(c) Show that for any \( x \in \mathbb{C}^n \), \( A^k x \to 0 \) as \( k \to \infty \).

(3) For each of the following IVP’s, compute the Picard iterates and identify the solution to which they converge.

(a) \( x' = tx, \quad x(0) = 1 \) \( (x \) scalar)\]

(b) \( x' = Ax, \quad x(0) = x_0 \) \( \text{where } A \in \mathbb{C}^{n \times n} \) is a constant matrix \( (x \in \mathbb{C}^n) \).

(4) (Gronwall’s Inequality — integral forms) Let \( \varphi, \psi, \alpha \) be real-valued continuous functions on the interval \( I = [a, b] \). Suppose \( \alpha \geq 0 \) on \( I \), and that

\[
\varphi(t) \leq \psi(t) + \int_a^t \alpha(s)\varphi(s)ds \quad \forall t \in I.
\]

(a) Show that

\[
\varphi(t) \leq \psi(t) + \int_a^t \exp \left( \int_s^t \alpha(r)dr \right) \alpha(s)\psi(s)ds \quad \forall t \in I.
\]

Hint: Let \( u(t) = \int_a^t \alpha(s)\varphi(s)ds \) and show that \( u' - \alpha u \leq \alpha \psi \).
(b) Suppose that \( \psi(t) \equiv c \) (a constant). Show that
\[
\varphi(t) \leq c \exp \left( \int_{a}^{t} \alpha(s)ds \right) \quad \forall t \in I.
\]

(5) (Iterative Methods for Linear Systems)

(a) Fix \( M \in \mathbb{C}^{n \times n} \) and \( g \in \mathbb{C}^{n} \). Given any \( x_0 \in \mathbb{C}^{n} \), define the sequence \( \{x_k\} \) iteratively by \( x_{k+1} = M x_k + g \). Show that if \( \rho(M) < 1 \), then \( I - M \) is invertible, and for any choice of \( x_0 \in \mathbb{C}^{n} \), \( x_k \to x^* \), the unique solution of \((I - M)x = g\).

(b) Suppose \( A \in \mathbb{C}^{n \times n} \) is invertible, \( b \in \mathbb{C}^{n} \) is given, and we want to solve the linear system \( Ax = b \) for \( x \in \mathbb{C}^{n} \). A splitting method writes \( A \) as \( A = S - T \) where \( S \) is invertible (and linear systems \( Sx = y \) are easily solved), and given \( x_0 \in \mathbb{C}^{n} \), define \( \{x_k\} \) by \( S x_{k+1} = T x_k + b \). Show that if \( \rho(S^{-1}T) < 1 \), then for any choice of \( x_0 \in \mathbb{C}^{n} \), \( x_k \to x^* \), the unique solution of \( Ax = b \).

(c) Suppose \( A \in \mathbb{C}^{n \times n} \) with nonzero diagonal entries. Write \( A = L + D + U \), where \( L \) is strictly lower triangular, \( D \) is diagonal, and \( U \) is strictly upper triangular. The Jacobi iteration is the splitting method where \( S = D, T = -(L + U) \). The Gauss-Seidel iteration is the splitting method where \( S = D + L, T = -U \). Show that if \( A \) is strictly (row) diagonally dominant (i.e., for \( 1 \leq i \leq n, |a_{ii}| > \sum_{j \neq i} |a_{ij}| \)), then \( A \) is invertible, and for any given \( x_0 \), the Jacobi iteration generates a sequence \( \{x_k\} \) which converges to the unique solution \( x^* \) of \( Ax = b \).

(Hint: Show \( \|D^{-1}(L + U)\|_\infty < 1 \) in the operator norm \( \|\cdot\|_\infty \) on \( \mathbb{C}^{n \times n} \) induced by the \( \ell^\infty \)-norm \( \|\cdot\|_\infty \) on \( \mathbb{C}^{n} \).)

[Remark: It can be shown that if \( A \) is strictly (row) diagonally dominant, then the Gauss-Seidel iteration converges. As one would expect from the proof of the Contraction Mapping Fixed Point Theorem, the rate of convergence depends on \( \rho(S^{-1}T) \). For some classes of matrices \( A \), it can be shown that \( \rho_G = \rho_J \), where \( \rho_G \) and \( \rho_J \) are \( \rho(S^{-1}T) \) for Gauss-Seidel and Jacobi, respectively, so Gauss-Seidel takes roughly half the number of iterations as Jacobi to achieve the same accuracy. With the goal of further decreasing \( \rho(S^{-1}T) \), Gauss-Seidel has been generalized to the successive over-relaxation method (SOR): the iteration takes the form
\[
(D + \omega L)x_{k+1} = D x_k + \omega [b - (D + U) x_k]
\]
where \( \omega \) (fixed) is called the relaxation parameter (\( \omega < 1 \) is called under-relaxation, \( \omega > 1 \) is called over-relaxation); the method was originally developed for matrices arising from discretizing elliptic PDE’s where values of \( \omega > 1 \) tend to give faster convergence, so the name SOR has stuck; Gauss-Seidel is SOR with \( \omega = 1 \). Dividing through by \( \omega \), SOR is seen to be a splitting method
\[
S_\omega x_{k+1} = T_\omega x_k + b,
\]
where
\[
S_\omega = \frac{1}{\omega}(D + \omega L) \quad \text{and} \quad T_\omega = \frac{1}{\omega}((1 - \omega)D - \omega U).
\]
The iteration matrix \( M \) (as in (a)) is
\[
M_\omega = S_\omega^{-1} T_\omega = (D + \omega L)^{-1}((1 - \omega)D - \omega U).
\]
It can be shown that if \( A \in \mathbb{R}^{n \times n} \) is symmetric positive-definite, then \( \rho(M_\omega) < 1 \) (and thus the SOR iteration converges) iff \( 0 < \omega < 2 \) (the Ostrowski-Reich Theorem). One direction (that \( 0 < \omega < 2 \) is necessary) is shown easily.]
(d) Suppose $A \in \mathbb{C}^{n \times n}$ with non-zero diagonal elements, $\omega \in \mathbb{R}$, and

$$M_\omega = (D + \omega L)^{-1}((1 - \omega)D - \omega U).$$

Show that $\rho(M_\omega) \geq |\omega - 1|$ (and thus $\rho(M_\omega) \geq 1$ for $\omega \leq 0$ and for $\omega \geq 2$).

(Hint: Use

$$\det (D + \omega L)^{-1} = \det D^{-1} \quad \text{and} \quad \det ((1 - \omega)D - \omega U) = \det ((1 - \omega)D)$$

to show that $\det (M_\omega) = (1 - \omega)^n$.)

(e) Let $A \in \mathbb{C}^{n \times n}$. A matrix $C \in \mathbb{C}^{n \times n}$ is called an approximate inverse for $A$ if $\rho(I - CA) < 1$. Show that if $C$ is an approximate inverse for $A$, then $A$ is invertible, $C$ is invertible, and for any given $x_0 \in \mathbb{C}^n$, the iteration $x_{k+1} = x_k + C(b - Ax_k)$ generates a sequence $\{x_k\}$ which converges to the unique solution $x_*$ of $Ax = b$.

[Remark: $r_k = b - Ax_k$ is called the residual at the $k$th iteration.]

(6) Use the Contraction Mapping Fixed Point Theorem to prove the following part of the Inverse Function Theorem:

If $\Phi : N \to \mathbb{R}^n$ is a $C^1$ mapping on a neighborhood $N \subset \mathbb{R}^n$ of $x_0 \in \mathbb{R}^n$ satisfying $\Phi(x_0) = y_0$ and $\Phi'(x_0) \in \mathbb{R}^{n \times n}$ is invertible, then $\Phi$ has a continuous right inverse $\Psi$ on some neighborhood of $y_0$.

Fill in the details of the following outline. Let $| \cdot |$ denote the Euclidean norm on $\mathbb{R}^n$ and $\| \cdot \|$ denote the induced operator norm on $\mathbb{R}^{n \times n}$. With $y \in \mathbb{R}^n$ as a parameter, define $G(x; y) = x + \Phi'(x_0)^{-1}(y - \Phi(x))$ for $x \in N$.

[Remark: Although $G$ is a nonlinear function of $x$, notice the similarity with problem 5(e). In fact, for $x$ near $x_0$, $\Phi'(x_0)^{-1}$ is an approximate inverse for $\Phi'(x)$.]

Let $G'(x; y)$ denote the Jacobian matrix of $G$ with respect to $x$ (for each fixed $y$).

(a) Show $\exists r > 0$ with $B_r(x_0) \subset N$ (where $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| \leq r\}$) for which

$$(\forall y \in \mathbb{R}^n) \ (\forall x \in B_r(x_0) \quad \|G'(x; y)\| \leq \frac{1}{2}).$$

(Hint: Show $G'(x; y)$ is independent of $y$, and $G'(x_0; y) = 0$.)

(b) Conclude that

$$(\forall y \in \mathbb{R}^n) \ (\forall x_1, x_2 \in B_r(x_0)) \quad |G(x_1; y) - G(x_2; y)| \leq \frac{1}{2}|x_1 - x_2|.$$

(c) Let

$$s = \frac{r}{2\|\Phi'(x_0)^{-1}\|}.$$ 

Show that for

$$y \in B_s(y_0) = \{y \in \mathbb{R}^n : |y - y_0| \leq s\},$$

$G(x; y)$ (as a function of $x$) maps $B_r(x_0)$ into itself.

(d) Show that for each $y \in B_s(y_0)$, $G(x; y)$ (as a function of $x$) has a unique fixed point $x_*$ (which we will call $\Psi(y)$) in $B_r(x_0)$. Show that the map $\Psi : B_s(y_0) \to B_r(x_0)$ satisfies

$$\Phi(\Psi(y)) = y \quad \forall \ y \in B_s(y_0).$$
(e) Show that $\Psi$ is continuous on $\overline{B_s(y_0)}$.

Hint: For each fixed $y \in B_s(y_0)$, let $G_k(x_0; y)$ denote the $k^{th}$ functional iterate of $G(x; y)$ (as a function of $x$) starting at $x_0$:

$$G_1(x_0; y) = G(x_0; y),$$

and for $k \geq 1$,

$$G_{k+1}(x_0; y) = G(G_k(x_0; y); y).$$

Show that as functions of $y \in B_s(y_0)$, each $G_k(x_0; y)$ is continuous, and $G_k(x_0; y)$ converges uniformly to $\Psi(y)$ on $\overline{B_s(y_0)}$. 