Math 554 Autumn 2014
Homework Set 8 Due Monday November 24

(1) (a) Let $f : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}$ be continuous. Suppose $x : \mathbb{R} \to \mathbb{C}$ is a solution of the n^{th}-order equation

$$(*) \qquad x^{(n)} = f(t, x, x', \vdots, x^{(n-1)}),$$

i.e., for each $t \in \mathbb{R}$, $x^{(n)}(t)$ exists and $x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t))$. Show that $x \in C^n(\mathbb{R})$.

(b) Define $F : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ by $F(t, y) = [y_2, y_3, \ldots, y_n, f(t, y_1, \ldots, y_n)]^T$ (so F is continuous). Suppose $y : \mathbb{R} \to \mathbb{C}^n$ is a solution of the first-order system

$$(**) \qquad y' = F(t, y),$$

i.e., for each $t \in \mathbb{R}$, $y'(t)$ exists and $y'(t) = F(t, y(t))$. Show that $y \in C^1(\mathbb{R})$, and moreover for $1 \le j \le n$, $y_j \in C^{n-j+1}(\mathbb{R})$.

(c) Show that if $x \in C^n(\mathbb{R})$ is a solution of $(*)$, then $y = [x, x', \ldots, x^{(n-1)}]^T$ is a C^1 solution of $(**)$. Moreover, if x satisfies the initial conditions

$$x^{(k)}(t_0) = x_0^k \qquad (0 \le k \le n-1),$$

then y satisfies the initial conditions $y(t_0) = [x_0^0, \ldots, x_0^{n-1}]^T$.

(d) Show that if y is a C^1 solution of $(**)$, then $x = y_1$ is a C^n solution of $(*)$. Moreover, if y satisfies the initial conditions $y(t_0) = y_0$, then x satisfies the initial conditions $x^{(k)}(t_0) = (y_0)_{k+1}$ $(0 \le k \le n-1)$.

(e) Show that the first-order system corresponding to the linear n^{th}-order equation $x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = b(t)$ is of the form $y' = A(t)y + B(t)$ where $A(t) \in \mathbb{C}^{n\times n}$ and $B(t) \in \mathbb{C}^n$, and identify $A(t)$ and $B(t)$.

(2) Let $A \in \mathbb{C}^{n\times n}$ have $\rho(A) < 1$.

(a) Show by example that A need not be a contraction with respect to the Euclidean metric.

(b) Show that there is an inner product on $\mathbb{C}^n$ so that A is a contraction with respect to the norm induced by the inner product.

(c) Show that for any $x \in \mathbb{C}^n$, $A^k x \to 0$ as $k \to \infty$.

(3) For each of th following IVP's, compute the Picard iterates and identify the solution to which they converge.

(a) $x' = tx, \qquad x(0) = 1 \qquad (x \text{ scalar})$

(b) $x' = Ax, \qquad x(0) = x_0 \qquad$ where $A \in \mathbb{C}^{n\times n}$ is a constant matrix $(x \in \mathbb{C}^n)$.

(4) (Gronwall's Inequality — integral forms) Let φ, ψ, α be real-valued continuous functions on the interval $I = [a, b]$. Suppose $\alpha \ge 0$ on I, and that

$$\varphi(t) \le \psi(t) + \int_a^t \alpha(s)\varphi(s)ds \quad \forall\, t \in I.$$

(a) Show that

$$\varphi(t) \le \psi(t) + \int_a^t \exp\left(\int_s^t \alpha(r)dr\right) \alpha(s)\psi(s)ds \quad \forall\, t \in I\ .$$

Hint: Let $u(t) = \int_a^t \alpha(s)\varphi(s)ds$ and show that $u' - \alpha u \le \alpha\psi$.

(b) Supposed that $\psi(t) \equiv c$ (a constant). Show that

$$\varphi(t) \leq c \exp\left(\int_a^t \alpha(s)ds\right) \quad \forall\, t \in I\ .$$

(5) (Iterative Methods for Linear Systems)

(a) Fix $M \in \mathbb{C}^{n\times n}$ and $g \in \mathbb{C}^n$. Given any $x_0 \in \mathbb{C}^n$, define the sequence $\{x_k\}$ iteratively by $x_{k+1} = Mx_k + g$. Show that if $\rho(M) < 1$, then $I - M$ is invertible, and for any choice of $x_0 \in \mathbb{C}^n$, $x_k \to x_*$, the unique solution of $(I - M)x = g$.

(b) Suppose $A \in \mathbb{C}^{n\times n}$ is invertible, $b \in \mathbb{C}^n$ is given, and we want to solve the linear system $Ax = b$ for $x \in \mathbb{C}^n$. A *splitting method* writes A as $A = S - T$ where S is invertible (and linear systems $Sx = y$ are easily solved), and given $x_0 \in \mathbb{C}^n$, define $\{x_k\}$ by $Sx_{k+1} = Tx_k + b$. Show that if $\rho(S^{-1}T) < 1$, then for any choice of $x_0 \in \mathbb{C}^n$, $x_k \to x_*$, the unique solution of $Ax = b$.

(c) Suppose $A \in \mathbb{C}^{n\times n}$ with nonzero diagonal entries. Write $A = L + D + U$, where L is strictly lower triangular, D is diagonal, and U is strictly upper triangular. The *Jacobi iteration* is the splitting method where $S = D$, $T = -(L+U)$. The *Gauss-Seidel iteration* is the splitting method where $S = D+L$, $T = -U$. Show that if A is *strictly (row) diagonally dominant* (i.e., for $1 \leq i \leq n$, $|a_{ii}| > \sum_{1\leq j\leq n, j\neq i} |a_{ij}|$), then A is invertible, and for any given x_0, the Jacobi iteration generates a sequence $\{x_k\}$ which converges to the unique solution x_* of $Ax = b$.
(Hint: Show $|||D^{-1}(L+U)|||_\infty < 1$ in the operator norm $|||\cdot|||_\infty$ on $\mathbb{C}^{n\times n}$ induced by the ℓ^∞-norm $\|\cdot\|_\infty$ on $\mathbb{C}^n$.)

[*Remark*: It can be shown that if A is strictly (row) diagonally dominant, then the Gauss-Seidel iteration converges. As one would expect from the proof of the Contraction Mapping Fixed Point Theorem, the rate of convergence depends on $\rho(S^{-1}T)$. For some classes of matrices A, it can be shown that $\rho_G = \rho_J^2$, where ρ_G and ρ_J are $\rho(S^{-1}T)$ for Gauss-Seidel an Jacobi, respectively, so Gauss-Seidel takes roughly half the number of iterations as Jacobi to achieve the same accuracy. With the goal of further decreasing $\rho(S^{-1}T)$, Gauss-Seidel has been generalized to the *successive over-relaxation method* (SOR): the iteration takes the form

$$(D + \omega L)x_{k+1} = D_{x_k} + \omega[b - (D + U)x_k]$$

where ω (fixed) is called the relaxation parameter ($\omega < 1$ is called under-relaxation, $\omega > 1$ is called over-relaxation); the method was originally developed for matrices arising from discretizing elliptic PDE's where values of $\omega > 1$ tend to give faster convergence, so the name SOR has stuck; Gauss-Seidel is SOR with $\omega = 1$. Dividing through by ω, SOR is seen to be a splitting method

$$S_\omega x_{k+1} = T_\omega x_k + b,$$

where

$$S_\omega = \frac{1}{\omega}(D + \omega L) \quad \text{and} \quad T_\omega = \frac{1}{\omega}((1-\omega)D - \omega U).$$

The iteration matrix M (as in (a)) is

$$M_\omega = S_\omega^{-1}T_\omega = (D + \omega L)^{-1}((1-\omega)D - \omega U).$$

It can be shown that if $A \in \mathbb{R}^{n\times n}$ is symmetric positive-definite, then $\rho(M_\omega) < 1$ (and thus the SOR iteration converges) iff $0 < \omega < 2$ (the Ostrowski-Reich Theorem). One direction (that $0 < \omega < 2$ is necessary) is shown easily.]

(d) Suppose $A \in \mathbb{C}^{n\times n}$ with nonzero diagonal elements, $\omega \in \mathbb{R}$, and

$$M_\omega = (D + \omega L)^{-1}((1-\omega)D - \omega U).$$

Show that $\rho(M_\omega) \geq |\omega - 1|$ (and thus $\rho(M_\omega) \geq 1$ for $\omega \leq 0$ and for $\omega \geq 2$). (Hint: Use

$$\det{(D + \omega L)^{-1}} = \det D^{-1} \quad \text{and} \quad \det{((1-\omega)D - \omega U)} = \det{((1-\omega)D)}$$

to show that $\det{(M_\omega)} = (1-\omega)^n$.)

(e) Let $A \in \mathbb{C}^{n\times n}$. A matrix $C \in \mathbb{C}^{n\times n}$ is called an *approximate inverse* for A if $\rho(I - CA) < 1$. Show that if C is an approximate inverse for A, then A is invertible, C is invertible, and for any given $x_0 \in \mathbb{C}^n$, the iteration $x_{k+1} = x_k + C(b - Ax_k)$ generates a sequence $\{x_k\}$ which converges to the unique solution x_* of $Ax = b$.

[*Remark*: $r_k = b - Ax_k$ is called the *residual* at the k^{th} iteration.]

(6) One-sided uniqueness theorem ($n = 1,\ \mathbb{F} = \mathbb{R}$)

(a) A real-valued function $f(t, u)$ is said to satisfy a one-sided Lipschitz condition in u if

$$u_1 > u_2 \quad \Rightarrow \quad f(t, u_1) - f(t, u_2) \leq L(u_2 - u_1) \quad \forall\ t \in \mathbb{R}.$$

Show that if f is continuous in t and u and satisfies a one-sided Lipschitz condition in u, then there is at most one solution to the IVP $u' = f(t, u),\ u(t_0) = 0$, for $t \geq t_0$.

(b) Let $f(t, u)$ be a real-valued continuous function in t and u, and suppose that f is decreasing in u for all t, i.e., $u_2 > u_1$ implies that $f(t, u_2) \leq f(t, u_1)$. Show that if $u(t)$ and $v(t)$ are both solutions to $u' = f(t, u)$, then

$$|u(t) - v(t)| \leq |u(s) - v(s)| \quad \text{whenever} \quad t \geq s\ .$$

Deduce uniqueness for the IVP $u' = f(t, u),\ u(t_0) = 0$, for $t \geq t_0$. Show, however, that uniqueness may fail for $t < t_0$.