Math 554 Autumn 2014
Homework Set 7 Due Monday November 17

(1) Let $A \in \mathbb{C}^{n\times n}$ have polar form $A = PU$. Show that A is normal iff $PU = UP$.

(2) Find the SVD for the matrix $A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 4 \end{bmatrix}$.

(3) Given $A \in \mathbb{C}^{m\times n}$ and $\epsilon > 0$, show that there exists $B \in \mathbb{C}^{m\times n}$ of full rank (i.e., $\operatorname{rank}(B) = \min(m, n)$) so that $\|A - B\| < \epsilon$, where $\|\cdot\|$ is the Euclidean operator norm.

(4) (a) Prove the following minimax characterization of the singular values $\sigma_1 \geq \cdots \geq \sigma_n$ of $A \in \mathbb{C}^{m\times n}$: for $1 \leq k \leq n$,

$$\sigma_k = \min_{S_{n-k+1}} \left(\max_{x\neq 0, x\in S_{n-k+1}} \frac{\|Ax\|}{\|x\|} \right),$$

where the min is taken over all subspaces S_{n-k+1} of dim. $n-k+1$, and $\|\cdot\|$ denotes the Euclidean norm. (Note: Here $\sigma_1 \geq \cdots \geq \sigma_n$, whereas in Courant-Fischer, we had $\lambda_1 \leq \cdots \leq \lambda_n$.)

(b) Use (a) to prove that if $A, B \in \mathbb{C}^{m\times n}$ have singular values $\sigma_1 \geq \cdots \geq \sigma_n$ and $\tau_1 \geq \cdots \geq \tau_n$ then for $1 \leq k \leq n$

$$|\sigma_k - \tau_k| \leq \|A - B\|,$$

where $\|\cdot\|$ is the Euclidean operator norm.

(c) Let $A \in \mathbb{C}^{m\times n}$ have rank $r > 1$, and suppose $1 \leq s < r$. Consider the problem of minimizing $\|A - B\|$ (Euclidean operator norm) over all matrices $B \in \mathbb{C}^{m\times n}$ of rank s. Show that the minimum value is σ_{s+1}, and identify a matrix B which achieves the min.

(5) Problem 4(b) shows that if a matrix is perturbed slightly, then its singular values can change at most by the (Euclidean operator) norm of the perturbation.

(a) Show that this result fails dramatically for eigenvalues in general by considering the perturbation

$$A_\epsilon = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & & & 1 \\ \epsilon & 0 & \cdots & 0 \end{bmatrix}$$

of $A_0 \in \mathbb{C}^n$. Find the eigenvalues of A_ϵ and compare with the eigenvalues of A_0 when ϵ is small. For example, let $n = 10$ and take $\epsilon = 10^{-10}$.

(b) Compute the singular values of A_ϵ and check that 4(b) really does hold in this case.

(6) (Cholesky Factorization) Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitain positive definite.

(a) Show that A can be written in factored form as $A = LDL^H$ where L is unit lower triangular and D is diagonal with positive entries. This factorization is called the *Cholesky factorization* of A. Here are some hints to get you going.

(i) Show that $A_{11} > 0$.

(ii) Let L_1 be the first Gaussian elimination matrix in our algorithm for computing the LU-factorization of A. Decribe the block structure of the matrix $L_1^{-1}AL_1^{-H}$.

(b) Show that a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ is positive semi-definite if and only if there is a matrix $B \in \mathbb{C}^{n\times n}$ such that $A = BB^H$.

(c) Suppose the Hermitian matrix A is positive semi-definite. Use the ideas behind Gaussian elimination with pivoting to construct a permutation matrix P, diagonal matrix D, and a lower triangular matrix L such that $A = PLDL^HP^H$.

(7) Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$ with associated eigenvector v. Let $b \in \mathbb{C}^n$ be any vector for which $\langle v, b\rangle \neq 0$. Consider solving the equation $(A - \lambda I)x = b$ numerically (i.e, subject to numerical error), and let $\bar{x}$ be the numerically computed solution. Give an arguement that explains why the vector $v = \bar{x}/\|\bar{x}\|$ is an excellent approximation to an eigenvector associated with the eigenvalue λ.
To illustrate this problem try the following Matlab commands:

$$\begin{aligned}
A &= 10 * (0.5 * ones(50,50) - rand(50,50));\\
ev &= eig(A);\\
b &= 10 * (0.5 * ones(50,1) - rand(50,1));\\
x &= (A - ev(1) * eye(50,50))\backslash b;\\
xn &= x/norm(x,2);\\
r &= (A - ev(1) * eye(50,50)) * xv;\\
& \quad norm(r)
\end{aligned}$$

The first command generates a random 50×50 matrix with entries between -10 and 10. The semicolon at the end of each line prevents Matlab from printing the results of the command line. The final line will print the 2-norm of the residual vector r. The norm of r should be on the order of 10^{-13} indicating that v is an eigenvector for the eigenvalue $ev(1)$ within the numerical precision of the floating point computations used in Matlab (14 decimal places). When you execute the command

$$x = (A - ev(1) * eye(50,50))\backslash b;$$

Matlab will send a warning message telling you about the near singularity of the matrix $(A - ev(1) * eye(50,50))$. The warning is to be expected in this case (why?) so just ignore it. What happens when you repeat the last four of the Matlab commands given above but with the vector b replaced by the eigenvector approximation v? In particular, what should happen to the new residual norm $\|r\|$?