Math 554  
Homework Set 7  
Autumn 2005  
Due Friday November 18

**Reading**  
Horn & Johnson, Sections 7.0–7.4, 7.7  
Skim through Horn & Johnson, Sections 6.0–6.3

(1) Let $A \in \mathbb{C}^{m \times n}$ have polar form $A = PU$. Show that $A$ is normal iff $PU = UP$.

(2) Find the SVD for the matrix $A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 4 \end{bmatrix}$.

(3) Given $A \in \mathbb{C}^{m \times n}$ and $\epsilon > 0$, show that there exists $B \in \mathbb{C}^{m \times n}$ of full rank (i.e., rank $(B) = \min(m, n)$) so that $\|A - B\| < \epsilon$, where $\| \cdot \|$ is the Euclidean operator norm.

(4) (a) Prove the following minimax characterization of the singular values $\sigma_1 \geq \cdots \geq \sigma_n$ of $A \in \mathbb{C}^{m \times n}$: for $1 \leq k \leq n$,

$$\sigma_k = \min_{S_{n-k+1}} \left( \max_{x \neq 0, x \in S_{n-k+1}} \frac{\|Ax\|}{\|x\|} \right),$$

where the min is taken over all subspaces $S_{n-k+1}$ of dim. $n-k+1$, and $\| \cdot \|$ denotes the Euclidean norm. (Note: Here $\sigma_1 \geq \cdots \geq \sigma_n$, whereas in Courant-Fischer, we had $\lambda_1 \leq \cdots \leq \lambda_n$.

(b) Use (a) to prove that if $A, B \in \mathbb{C}^{m \times n}$ have singular values $\sigma_1 \geq \cdots \geq \sigma_n$ and $\tau_1 \geq \cdots \geq \tau_n$ then for $1 \leq k \leq n$

$$|\sigma_k - \tau_k| \leq \|A - B\|,$$

where $\| \cdot \|$ is the Euclidean operator norm.

(c) Let $A \in \mathbb{C}^{m \times n}$ have rank $r > 1$, and suppose $1 \leq s < r$. Consider the problem of minimizing $\|A - B\|$ (Euclidean operator norm) over all matrices $B \in \mathbb{C}^{m \times n}$ of rank $s$. Show that the minimum value is $\sigma_{s+1}$, and identify a matrix $B$ which achieves the min.

(5) Problem 4(b) shows that if a matrix is perturbed slightly, then its singular values can change at most by the (Euclidean operator) norm of the perturbation.

(a) Show that this result fails drastically for eigenvalues in general by considering the perturbation

$$A_\epsilon = \begin{bmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 1 \\ \epsilon & 0 & \cdots & 0 \end{bmatrix}$$

of $A_0 \in \mathbb{C}^n$. Find the eigenvalues of $A_\epsilon$ and compare with the eigenvalues of $A_0$ when $\epsilon$ is small. For example, let $n = 10$ and take $\epsilon = 10^{-10}$.

(b) Compute the singular values of $A_\epsilon$ and check that 4(b) really does hold in this case.

(6) (a) Use the LU-factorization to compute bases for the four fundamental subspaces associated with the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 8 & 4 \\ -1 & 0 & 1 & -2 & -8 & -3 \\ 2 & 2 & 5 & 1 & 15 & 7 \\ 1 & 2 & 6 & -1 & 7 & 4 \end{bmatrix}.$$
(b) Use the solution to part (a) to completely describe the set of solutions to the system
\[ Ax = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}. \]

(7) (Cholesky Factorization) Suppose \( A \in \mathbb{C}^{n \times n} \) is Hermitian positive definite.

(a) Show that \( A \) can be written in factored form as \( A = LDL^H \) where \( L \) is unit lower triangular and \( D \) is diagonal with positive entries. This factorization is called the Cholesky factorization of \( A \). Here are some hints to get you going.

(i) Show that \( A_{11} > 0 \).

(ii) Let \( L_1 \) be the first Gaussian elimination matrix in our algorithm for computing the LU-factorization of \( A \). Describe the block structure of the matrix \( L_1^{-1}AL_1^{-H} \).

(b) Show that a Hermitian matrix \( A \in \mathbb{C}^{n \times n} \) is positive semi-definite if and only if there is a matrix \( B \in \mathbb{C}^{n \times n} \) such that \( A = BB^H \).

(c) Suppose the Hermitian matrix \( A \) is positive semi-definite. Use the ideas behind Gaussian elimination with pivoting to construct a permutation matrix \( P \), diagonal matrix \( D \), and a lower triangular matrix \( L \) such that \( A = PLDL^H P^H \).