Math 554 | Autumn 2014
Homework Set 6 | Due Friday, November 7

Reading Horn & Johnson, Chapter 4 - 5

(1) Show that $U \in \mathbb{C}^{n\times n}$ is unitary iff $\exists$ a Hermitian $H \in \mathbb{C}^{n\times n}$ for which $U = e^{iH}$.

(2) Use the Schur Triangularization Theorem to show that every matrix $A \in \mathbb{C}^{n\times n}$ is "almost" diagonalizable in the following two senses.
  (a) Given $\epsilon > 0$, there is a matrix $\widetilde{A} \in \mathbb{C}^{n\times n}$ with distinct eigenvalues for which $\|A - \widetilde{A}\|_F < \epsilon$.
  (b) Given $\epsilon > 0$, there is an upper triangular T similar to A for which $|t_{ij}| < \epsilon$ for all $i < j$.

(3) Let $A, B \in \mathbb{C}^{n\times n}$ be diagonalizable. Show that A and B are *simultaneously diagonalizable* (i.e., $\exists$ one invertible $S \in \mathbb{C}^{n\times n}$ for which both $S^{-1}AS$ and $S^{-1}BS$ are diagonal) iff $AB = BA$, as follows:
  (a) Show that if A, B are simultaneously diagonalizable, then $AB = BA$.
  (b) Suppose $AB = BA$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A, with eigenspaces $E_1, \ldots, E_k$ and associated projections $P_1, \ldots, P_k$. Show that $BE_i \subset E_i$ for each i and deduce that $BP_i = P_iB$ for each i.
  (c) Suppose $AB = BA$. Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbb{C}^n$ consisting of eigenvectors of B. Show that for each i, $1 \le i \le i$, the vectors $\{P_iv_1, \ldots, P_iv_n\}$ span the subspace E_i (where E_i, P_i are as in part (b)). Also show that each nonzero P_iv_j is an eigenvector of B.
  (d) Suppose $AB = BA$. Deduce that there is a basis for $\mathbb{C}^n$ consisting of vectors which are eigenvectors for both A and B. Conclude that A and B are simultaneously diagonalizable.

(4) Prove the real version of the Schur Theorem: If $A \in \mathbb{R}^{n\times n}$, there is an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ so that $V^TAV = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{bmatrix} \equiv B$ is block upper triangular, where each A_i is either a real 1×1 matrix or a real 2×2 matrix with eigenvalues a complex conjugate pair $\lambda \neq \bar{\lambda}$. (The matrix $B \in \mathbb{R}^{n\times n}$ is often called *quasi-upper-triangular*.)

(5) (a) What are the possible Jordan canonical forms for A if $p_A(t) = (t+3)^4(t-4)^2$?
  (b) Find the Jordan canonical form for $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

(6) Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and inner product norm $\|\cdot\|$. A set $K \subset \mathbb{H}$ is said to be a cone if $\alpha K \subset K$ for all $\alpha > 0$. Given a set $C \subset \mathbb{H}$, define the *polar* of C to be the set $C^\circ := \{\, y \in \mathbb{H} \mid \langle y, x \rangle \le 1 \ \forall x \in C \,\}$.
  (a) Show that $K \subset \mathbb{H}$ is a convex cone if and only if (i) $\alpha K \subset K$ for all $\alpha > 0$ and (ii) $K + K \subset K$.
  (b) Let $\mathbb{B}$ be the closed unit ball in $\mathbb{H}$. Show that $\mathbb{B}^\circ = \mathbb{B}$.
  (c) Given $C \subset \mathbb{H}$, show that C° is always a closed convex set.
  (d) Given $C \subset \mathbb{H}$, show that $(C^\circ)^\circ$ is the closed convex hull of C, i.e., the intersection of all closed convex sets containing C.

(e) Given a non-empty closed convex cone $K \subset \mathbb{H}$ show that

$$K^\circ := \{\, y \in \mathbb{H} \mid \langle y,\, x\rangle \leq 0 \;\forall\, x \in K \,\}\,.$$

(f) Let $S \subset \mathbb{H}$ be a closed subspace. Observe that S is a closed convex cone. Show that $S^\circ = S^\perp$.

(g) Let $K \subset \mathbb{H}$ be a non-empty closed convex cone. Given $z, x, y \in \mathbb{H}$, show that the following statements are equivalent:

(i) $z = x + y$ with $x \in K,\ y \in K^\circ$ and $\langle x,\, y\rangle = 0$.

(ii) $x = P_K z$ and $y = P_{K^\circ} z$, where P_K and P_{K° are the projections onto K and K°, respectively.