Math 554 — Autumn 2014

Homework Set 5 — Due Friday October 31

Reading Horn & Johnson, Chapter 1 - 3

(1) (a) Let $A \in \mathbb{C}^{n\times n}$ be invertible and set $|\lambda|_{\max} = \max\{|\lambda| : \lambda \in \Sigma(A)\}$, $|\lambda|_{\min} = \min\{|\lambda| : \lambda \in \Sigma(A)\}$. Show that the condition number of A relative to any submultiplicative norm satisfies $k(A) \geq |\lambda|_{\max}/|\lambda|_{\min}$. (So if this ratio is large, A is ill-conditioned.)

(b) Let $A, B \in \mathbb{C}^{n\times n}$ with A invertible and B singular. Show that the condition number of A relative to any submultiplicative norm $\|\cdot\|$ satisfies $k(A) \geq \frac{\|A\|}{\|A-B\|}$.

(c) Use part (b) to show that if $A \in \mathbb{C}^{n\times n}$ is upper triangular and invertible, then the condition number of A relative to $|||\cdot|||_\infty$ satisfies $k(A) \geq \frac{|||A|||_\infty}{\min_{1\leq i\leq n}|a_{ii}|}$.

(2) Let $A \in \mathbb{C}^{n\times n}$ and suppose that λ is an eigenvalue of A of algebraic multiplicity 1. Show that rank $(A - \lambda I) = n - 1$, but not conversely.

(3) (a) Let $A \in \mathbb{C}^{n\times n}$ be normal and suppose all eigenvalues of A are real. Show that A is Hermitian.

(b) Let $A \in \mathbb{C}^{n\times n}$ be normal and suppose all eigenvalues of A satisfy $|\lambda| = 1$. Show that A is unitary.

(c) Show by examples that (a) and (b) both fail if the normality assumption is reduced to assuming just that A is diagonalizable.

(4) (a) Let $A, B \in \mathbb{C}^{n\times n}$ and suppose A is invertible. Show that AB and BA are similar. Show by example that this need not be true if A and B are both allowed to be singular.

(b) Let $A, B \in \mathbb{C}^{n\times n}$ show that $A + \epsilon I$ is invertible for all sufficiently small $\epsilon > 0$. Use part (a) and take limits to show that for any $A, B \in \mathbb{C}^{n\times n}$, AB and BA have the same characteristic polynomials. Deduce that AB and BA have the same eigenvalues, including algebraic multiplicities.

(5) If $A, B \in \mathbb{C}^{n\times n}$ are Hermitian, we say that $A \geq B$ if $A - B \in \mathcal{H}_+^n$, where $\mathcal{H}_+^n$ is the set of Hermitian positive semi-definite matrices.

(a) Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and $\alpha \geq 0$. Show that $\|A\| \leq \alpha$ iff $-\alpha I \leq A \leq \alpha I$ (where $\|\cdot\|$ is the operator 2-norm).

(b) If A is Hermitian and $\alpha I \leq A \leq \beta I$, and if p is a polynomial for which $p \geq 0$ on $[\alpha, \beta]$, show that $p(A) \geq 0$.

(c) If A is Hermitian and $\alpha I \leq A \leq \beta I$, and if p is a polynomial such that $p \neq 0$ on $[\alpha, \beta]$, show that $p(A)$ is invertible.

(6) The following describes an iterative procedure, called the *power method*, for (usually) finding the largest eigenvalue (in magnitude) and an associated eigenvector for $A \in \mathbb{C}^{n\times n}$. Although the method applies in a more general setting, assume for simplicity that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A \in \mathbb{C}^{n\times n}$ satisfy $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$ and that A is diagonalizable. Given $x_0 \in \mathbb{C}^n$, define a sequence $\{x_k\}$ in $\mathbb{C}^n$ inductively by $x_{k+1} = \frac{1}{\|x_k\|_2} A x_k$. Show that for x_0 in a dense open subset in $\mathbb{C}^n$ (actually, the complement of a hyperplane—identify this set explicitly), the generated sequence $\{x_k\}$ converges to an eigenvector of A corresponding to λ_1. Show also that for $1 \leq i \leq n$, if the i^{th} component of this eigenvector is nonzero, then $\lim_{k\to\infty}(Ax_k)_i/(x_k)_i = \lambda_1$.