Math 554 Autumn 2014
Homework Set 4 Due Friday October 24

Reading Horn & Johnson, Chapter 1

(1) (a) Find the best constants m, M so that $m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$ for $x \in \mathbb{C}^n$.
(b) Find the best constants m, M so that $m|||A|||_1 \le |||A|||_2 \le M|||A|||_1$ for $A \in \mathbb{C}^{n\times n}$.
(c) For $A \in \mathbb{C}^{n\times n}$, find the best constants m, M so that $m|||A|||_1 \le \|A\|_F \le M|||A|||_1$, where $\|A\|_F = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}}$ is the Frobenius (Hilbert-Schmidt) norm.

(2) It is shown in the notes that a linear functional $f \in \mathbb{F}^{n^*}\backslash\{0\}$ is in the closed unit ball of the dual space $\mathbb{F}_n^*$ with respect to the dual norm of a norm $\|\cdot\|$ on $\mathbb{F}^n$ if and only if the real hyperplane $\{x : \mathcal{R}ef(x) = 1\}$ corresponding to f lies completely outside the open unit ball $\{x : \|x\| < 1\}$ of $\|\cdot\|$. Use this criterion to give a proof that the dual norm to the ℓ^∞-norm on $\mathbb{R}^2$ is the ℓ^1-norm by explicitly identifying the dual unit ball.

(3) Let $(V, \|\cdot\|)$ be a Banach space.
(a) Show that if $L \in \mathcal{B}(V)$ and the operator norm $\|L\|$ satisfies $\|L\| < 1$, then $I - L$ is invertible and $\frac{1}{1+\|L\|} \le \|(I-L)^{-1}\| \le \frac{1}{1-\|L\|}$.
(b) An operator $L \in \mathcal{B}(V)$ is called *invertible* if L is bijective and $L^{-1} \in \mathcal{B}(V)$. Show that if L is invertible, then the open ball in $\mathcal{B}(V)$ about L of radius $\frac{1}{\|L^{-1}\|}$ consists entirely of invertible operators. (So the set of invertible operators in $\mathcal{B}(V)$ is *open*.)

(4) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Identify $\mathbb{R}^n$ with its dual using the usual Euclidean inner product, so the dual norm is $\|x\|^* = \sup_{\|y\|\le 1} |\langle x, y\rangle|$ for $x \in \mathbb{R}^n$. Show that $\left(\sup_{\|x\|=1} \|x\|_2\right)^2 = \sup_{\|x\|=1} \|x\|^*$ and $\left(\inf_{\|x\|=1} \|x\|_2\right)^2 = \inf_{\|x\|=1} \|x\|^*$. Conclude that $\|\cdot\| = \|\cdot\|^* \Leftrightarrow \|\cdot\| = \|\cdot\|_2$.

(5) If $A \in \mathbb{C}^{n\times n}$, recall that λ is an eigenvalue for A if there is a nonzero $x \in \mathbb{C}^n$ with $Ax = \lambda x$.
(a) If $\|\cdot\|$ is any submultiplicative norm on $\mathbb{C}^{n\times n}$ and λ is an eigenvalue of A, show that $\|\lambda\| \le \|A\|$.
(b) If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a monic polynomial with coefficients $a_i \in \mathbb{C}$, define the *companion matrix* of p to be

$$C(p) = \begin{bmatrix} 0 & 1 & & \bigcirc \\ & \ddots & \ddots & \\ \bigcirc & & & \\ & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}.$$

Show that the eigenvalues of $C(p)$ are precisely the zeroes of p.
(c) Take $\|\cdot\| = |||\cdot|||_1$ and $A = C(p)$ in part (a) to deduce *Cauchy's bound*: all roots z of p satisfy $|z| \le 1 + \max\{|a_0|, \ldots, |a_{n-1}|\}$.
(d) Use $\|\cdot\| = |||\cdot|||_\infty$ and deduce *Montel's bound*: $|z| \le \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}$ for roots z of p.
(e) Apply (d) to the polynomial $(z-1)p(z)$ to deduce another bound of Montel: $|z| \le |a_0| + |a_1 - a_0| + \cdots + |a_{n-1} - 1|$.

(f) Use (3) to prove *Kakeya's Theorem*: if $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial with real nonnegative coefficients satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0$, then all roots of f lie in the closed unit disk in $\mathbb{C}$.