Math 554 Autumn 2014
Homework Set 3 Due Friday October 17

Reading Horn & Johnson, Chapter 5
Continue your review of linear algebra

(1) (The parallelogram law is a necessary and sufficient condition for a norm to be induced by an inner product.)
   (a) Show that if a norm $\|\cdot\|$ on a vector space V comes from an inner product, then $\|\cdot\|$ satisfies the parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.
   (b) Let $\|\cdot\|$ be the norm induced by an inner product $\langle\cdot,\cdot\rangle$ on a vector space V. Show the polarization identity
      (i) if $\mathbb{F} = \mathbb{R}$, $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$
      (ii) if $\mathbb{F} = \mathbb{C}$, $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$
   (Remark: The polarization identity expresses an inner product in terms of the norm that it induces. The converse of part (a) is true as well: if $\|\cdot\|$ is a norm satisfying the parallelogram law, then it is 8induced by an inner product. If you want a challenge, try to prove this. Start by using the polarization identity to define $\langle x, y\rangle$, and then use the parallelogram law to show that this really defines an inner product which induces $\|\cdot\|$.)
   (c) Is $C[a, b]$ with norm $\|u\| = \sup_{[a,b]} |u(x)|$ an inner product space?
   (d) Show that ℓ^2 is the only inner product space among the ℓ^p spaces, $1 \leq p \leq \infty$.

(2) A sequence $\{v_n\}$ in an inner product space $(V, \langle\cdot,\cdot\rangle)$ is said to *converge weakly* to v if $(\forall\, w \in V)\langle v_n, w\rangle \to \langle v, w\rangle$, and is said to *converge strongly* to v if $\|v_n - v\| \to 0$.
   (a) Show that if $\dim V < \infty$ and $v_n \to v$ weakly, then $v_n \to v$ strongly.
   (b) Show that part (a) fails for $V = \ell^2$.
   (c) Show that for any $(V, \langle\cdot,\cdot\rangle)$, if $v_n \to v$ strongly, then $v_n \to v$ weakly.

(3) (a) Show that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$, where $\|x\|_p$ denotes the ℓ^p-norm of $x \in \mathbb{C}^n$.
   (b) Show that $\|u\|_\infty = \lim_{p\to\infty} \|u\|_p$, where $\|u\|_p$ denotes the L^p-norm of $u \in C[a, b]$.

(4) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V. Suppose that all sequences $\{v_n\} \subset V$ which satisfy $\|v_n\|_1 \to 0$ also satisfy $\|v_n\|_2 \to 0$, and vice-versa. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.

(5) Let $C^1[a, b]$ be the space of continuous functions on $[a, b]$ whose derivative exists at each point of $[a, b]$ (one-sided derivative at the endpoints) and defines a continuous function on $[a, b]$.
   (a) Suppose $u \in C([a, b]) \cap C^1((a, b))$ and u' extends continuously to $[a, b]$. Show that $u \in C^1([a, b])$.
   (b) Show that $\|u\| = \sup_{[a,b]} |u(x)| + \sup_{[a,b]} |u'(x)|$ is a norm on $C^1[a, b]$ which makes $C^1[a, b]$ into a Banach space. (Hint for completeness: if $u_n \in C^1[a, b]$ satisfies $u_n \to u$ uniformly and $u_n' \to v$ uniformly, take limits in the equation $u_n(x) - u_n(a) = \int_a^x u_n'(s)ds$ to show $u \in C^1[a, b]$ and $u' = v$.)
   (c) Sketch how the arguments in parts (a) and (b) extend to $C^k[a, b]$: $\|u\| = \sup|u| + \sup|u'| + \cdots + \sup|u^{(k)}|$ makes $C^k[a, b]$ into a Banach space (where k is an integer, $k \geq 1$).

(6) If $0 < \alpha \leq 1$, a function $u \in C[a, b]$ is said to satisfy a Hölder condition of order α (or to be Hölder continuous of order α) if $\sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty$. Denote by $\Lambda^\alpha([a, b])$ this set of functions.

(a) Show that $\|u\|_\alpha = \sup |u| + \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$ is a norm which makes $\Lambda^\alpha([a,b])$ into a Banach space.

(b) Show that $C^1([a,b]) \subset \Lambda^\alpha([a,b])$, and the inclusion map is continuous with respect to the norms defined above.

(Remark: A function $f \in \Lambda^\alpha([a,b])$ for $\alpha = 1$ is called Lipschitz continuous.)

(7) Consider the linear operator $T : L^2((0,1)) \to L^2((0,1))$ defined by

$$Tf(y) = \int_0^1 (y-x) f(x) dx.$$

(a) Describe the range and null space of T.

(b) Derive an explicit expression for T^T as a mapping from $L^2((0,1))$ to itself.

(c) Show that T is normal, i.e. $T^T T = T T^T$.