Reading  Horn & Johnson, Chapter 5
Continue your review of linear algebra

(1) (The parallelogram law is a necessary and sufficient condition for a norm to be induced by an inner product.)
(a) Show that if a norm \( \| \cdot \| \) on a vector space \( V \) comes from an inner product, then
\[ \| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2). \]
(b) Let \( \| \cdot \| \) be the norm induced by an inner product \( \langle \cdot , \cdot \rangle \) on a vector space \( V \). Show the polarization identity
\[
\begin{align*}
(\text{i}) & \quad \text{if } \mathbb{F} = \mathbb{R}, \quad \langle x, y \rangle = \frac{1}{2}(\| x + y \|^2 - \| x - y \|^2) \\
(\text{ii}) & \quad \text{if } \mathbb{F} = \mathbb{C}, \quad \langle x, y \rangle = \frac{1}{2}(\| x + y \|^2 - \| x - y \|^2 + i\| x + iy \|^2 - i\| x - iy \|^2)
\end{align*}
\]
(Remark: The polarization identity expresses an inner product in terms of the norm that it induces. The converse of part (a) is true as well: if \( \| \cdot \| \) is a norm satisfying the parallelogram law, then it is induced by an inner product. If you want a challenge, try to prove this. Start by using the polarization identity to define \( \langle x, y \rangle \), and then use the parallelogram law to show that this really defines an inner product which induces \( \| \cdot \| \).)
(c) Is \( C[a,b] \) with norm \( \| u \| = \sup_{[a,b]} |u(x)| \) an inner product space?
(d) Show that \( \ell^2 \) is the only inner product space among the \( \ell^p \) spaces, \( 1 \leq p \leq \infty \).

(2) A sequence \( \{v_n\} \) in an inner product space \( (V, \langle \cdot , \cdot \rangle) \) is said to converge weakly to \( v \) if \( \langle v_n, w \rangle \to \langle v, w \rangle \) for all \( \forall w \in V \), and is said to converge strongly to \( v \) if \( \| v_n - v \| \to 0 \).
(a) Show that if \( \dim V < \infty \) and \( v_n \to v \) weakly, then \( v_n \to v \) strongly.
(b) Show that part (a) fails for \( V = \ell^2 \).
(c) Show that for any \( (V, \langle \cdot , \cdot \rangle) \), if \( v_n \to v \) strongly, then \( v_n \to v \) weakly.
(d) Show that \( \| x \|_\infty = \lim_{p \to \infty} \| x \|_p \), where \( \| x \|_p \) denotes the \( \ell^p \)-norm of \( x \) in \( \mathbb{C}^n \).
(e) Show that \( \| u \|_\infty = \lim_{p \to \infty} \| u \|_p \), where \( \| u \|_p \) denotes the \( L^p \)-norm of \( u \) in \( C[a,b] \).

(4) Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be two norms on a vector space \( V \). Suppose that all sequences \( \{v_n\} \subset V \) which satisfy \( \| v_n \|_1 \to 0 \) also satisfy \( \| v_n \|_2 \to 0 \), and vice-versa. Show that \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent norms.

(5) Let \( C^1[a,b] \) be the space of continuous functions on \( [a,b] \) whose derivative exists at each point of \( [a,b] \) (one-sided derivative at the endpoints) and defines a continuous function on \( [a,b] \).
(a) Suppose \( u \in C([a,b]) \cap C^1((a,b)) \) and \( u' \) extends continuously to \( [a,b] \). Show that \( u \in C^1([a,b]) \).
(b) Show that \( \| u \| = \sup_{[a,b]} |u(x)| + \sup_{[a,b]} |u'(x)| \) is a norm on \( C^1[a,b] \) which makes \( C^1[a,b] \) into a Banach space. (Hint for completeness: if \( u_n \in C^1[a,b] \) satisfies \( u_n \to u \) uniformly and \( u'_n \to v \) uniformly, take limits in the equation \( u_n(x) - u_n(a) = \int_a^x u'_n(s)ds \) to show \( u \in C^1[a,b] \) and \( u' = v \).
(c) Sketch how the arguments in parts (a) and (b) extend to \( C^k[a,b] : \| u \| = \sup |u| + \sup |u'| + \cdots + \sup |u^{(k)}| \) makes \( C^k[a,b] \) into a Banach space (where \( k \) is an integer, \( k \geq 1 \)).

(6) If \( 0 < \alpha \leq 1 \), a function \( u \in C[a,b] \) is said to satisfy a Hölder condition of order \( \alpha \) (or to be Hölder continuous of order \( \alpha \)) if \( \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \). Denote by \( \Lambda^\alpha([a,b]) \) this set of functions.
(a) Show that \( \|u\|_\alpha = \sup |u| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \) is a norm which makes \( \Lambda^\alpha([a, b]) \) into a Banach space.

(b) Show that \( C^1([a, b]) \subset \Lambda^\alpha([a, b]) \), and the inclusion map is continuous with respect to the norms defined above.

(Remark: A function \( f \in \Lambda^\alpha([a, b]) \) for \( \alpha = 1 \) is called Lipschitz continuous.)

(7) Consider the linear operator \( T : L^2((0, 1)) \to L^2((0, 1)) \) defined by

\[
Tf(y) = \int_0^1 (y - x)f(x)\,dx.
\]

(a) Describe the range and null space of \( T \).

(b) Derive an explicit expression for \( T^T \) as a mapping from \( L^2((0, 1)) \) to itself.

(c) Show that \( T \) is normal, i.e. \( T^TT = TT^T \).