Math 554 | Autumn 2014
Homework Set 2 | Due Friday October 10

Reading Continue your review of linear algebra, including determinants in H-J.

(1) Define subspaces $W_1, W_2 \subset C[0,1]$ by $W_1 = \{f : \int_0^1 f(x)dx = 0\}$ and $W_2 = \{f : f(x) = c$ for some $c \in \mathbb{C}$ and all $x \in [0,1]\}$. Show that $C[0,1] = W_1 \oplus W_2$, and derive explicit formulas for the projection operators P_1, P_2 onto W_1, W_2.

(2) Let V, W be finite-dimensional vector spaces of dimension n, m, respectively. Let $L \in \mathcal{B}(V,W)$, and suppose rank $(L) = 1$. Show that for any choice of bases for V and W, the matrix of L is of the form ab^T where $a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. (This matrix is called the *outer product* of the vectors a and b.)

(3) Let $\mathcal{P}_n$ be the space of polynomials of degree n or less, and consider the differential operator D as a mapping from $\mathcal{P}_n$ to $\mathcal{P}_n$ ($D : \mathcal{P}_n \mapsto \mathcal{P}_n$).
   (a) Show that D is nilpotent.
   (b) Provide at least two different bases in which the matrix representation for D in these bases is a direct product of shift operators.

(4) (In this problem, take $\mathbb{F} = \mathbb{R}$.) Let $A \in \mathbb{R}^{n\times n}$. One can regard the entries a_{ij} as independent variables, and $\det A$ as a function of these n^2 variables.
   (a) Show that $\frac{\partial}{\partial a_{ij}}(\det A) = \widehat{A}_{ij}$, where $\widehat{A}_{ij}$ is the (i,j) cofactor of A, i.e., $\widehat{A}_{ij} = (-1)^{i+j}\det(A[i|j])$, where $A[i|j]$ is the $(n-1)\times(n-1)$ submatrix of A obtained by removing the i^{th} row and j^{th} column of A.
   (b) Suppose $A(t)$ is an $\mathbb{R}^{n\times n}$-valued function of $t \in \mathbb{R}$ whose entries $a_{ij}(t)$ are differentiable functions of t. Show that $\frac{d}{dt}(\det A(t)) = \sum_{i=1}^n \sum_{j=1}^n \widehat{A}_{ij}(t)\frac{d}{dt}(a_{ij}(t))$.
   (c) Suppose in addition $A(0) = I$. Show that $\frac{d}{dt}(\det A(t))\Big|_{t=0} = \operatorname{tr}\left(\frac{dA}{dt}(0)\right)$.
   (d) Suppose $A(t)$ is as in part (b), and suppose $\det A(t) \neq 0$ for all $t \in \mathbb{R}$. Show $\frac{d}{dt}\log(\det A(t)) = \sum_{i=1}^n \sum_{j=1}^n (A^{-1})_{ji}\frac{d}{dt}(a_{ij}(t))$, where $(A^{-1})_{ji}$ is the ji^{th} entry of $A(t)^{-1}$.

(5) This problem continues and extends problem 2 on Problem Set 1.
   (a) (extension of 2(b) on P.S.1) Let $\{a_0, a_1, \ldots, a_n\}$ be distinct real numbers. Show that the linear functionals $f_0, \ldots, f_n$ on $\mathcal{P}_n$ defined by $f_k(p) = p(a_k)$ form a basis of $\mathcal{P}_n^*$.
   (b) Define polynomials $\ell_0, \ldots, \ell_n \in \mathcal{P}_n$ by $\ell_k(x) = \prod_{0\le i\le n, i\ne k}\left(\frac{x-a_i}{a_k-a_i}\right)$. Show that $\ell_0, \ldots, \ell_n$ are linearly independent, and thus form a basis of $\mathcal{P}_n$. (Hint: first show $\ell_k(a_i) = \delta_{ik}$.) The ℓ_k's are called the *Lagrange polynomials* (associated with the set $\{a_0, \ldots, a_n\}$).
   (c) Show that $\{f_0, \ldots, f_n\}$ in part (a) form the basis of $\mathcal{P}_n^*$ which is dual to the basis $\{\ell_0, \ldots, \ell_n\}$ of $\mathcal{P}_n$.
   (d) Show that if $c_0, \ldots, c_n \in \mathbb{C}$, then $p(x) = \sum_{k=0}^n c_k\ell_k(x)$ is the unique element of $\mathcal{P}_n$ satisfying the interpolation conditions $p(a_k) = c_k$, $0 \le k \le n$. This result is called the Lagrange interpolation formula. It provides an explicit formula for

a polynomial p of degree at most n having prescribed values $c_0, \ldots, c_n$ at the prescribed distinct points $a_0, \ldots, a_n$.

(6) Formulate and prove a precise statement of the following fact:

If $\{v_1, \ldots, v_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ (clearly $k \leq n$), and each of $v_1, \ldots, v_k$ is perturbed slightly, then the resulting set of vectors is also linearly independent.