Reading  Continue your review of linear algebra, including determinants in H-J.

(1) Define subspaces $W_1, W_2 \subset C[0,1]$ by $W_1 = \{ f : \int_0^1 f(x)dx = 0 \}$ and $W_2 = \{ f : f(x) = c \text{ for some } c \in \mathbb{C} \text{ and all } x \in [0,1] \}$. Show that $C[0,1] = W_1 \oplus W_2$, and derive explicit formulas for the projection operators $P_1, P_2$ onto $W_1, W_2$.

(2) Let $V, W$ be finite-dimensional vector spaces of dimension $n, m$, respectively. Let $L \in B(V, W)$, and suppose rank $(L) = 1$. Show that for any choice of bases for $V$ and $W$, the matrix of $L$ is of the form $ab^T$ where $a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. (This matrix is called the outer product of the vectors $a$ and $b$.)

(3) Let $P_n$ be the space of polynomials of degree $n$ or less, and consider the differential operator $D$ as a mapping from $P_n$ to $P_n$ ($D : P_n \mapsto P_n$).

(a) Show that $D$ is nilpotent.
(b) Provide at least two different bases in which the matrix representation for $D$ in these bases is a direct product of shift operators.

(4) (In this problem, take $\mathbb{F} = \mathbb{R}$.) Let $A \in \mathbb{R}^{n \times n}$. One can regard the entries $a_{ij}$ as independent variables, and $\det A$ as a function of these $n^2$ variables.

(a) Show that $\frac{\partial}{\partial a_{ij}}(\det A) = \hat{A}_{ij}$, where $\hat{A}_{ij}$ is the $(i,j)$ cofactor of $A$, i.e., $\hat{A}_{ij} = (-1)^{i+j}\det(A[i,j])$, where $A[i,j]$ is the $(n-1) \times (n-1)$ submatrix of $A$ obtained by removing the $i^{th}$ row and $j^{th}$ column of $A$.
(b) Suppose $A(t)$ is an $\mathbb{R}^{n \times n}$-valued function of $t \in \mathbb{R}$ whose entries $a_{ij}(t)$ are differentiable functions of $t$. Show that $\frac{d}{dt}(\det A(t)) = \sum_{i=1}^n \sum_{j=1}^n \hat{A}_{ij}(t) \frac{d}{dt}(a_{ij}(t))$.

(c) Suppose in addition $A(0) = I$. Show that $\frac{d}{dt}(\det A(t)) \bigg|_{t=0} = \text{tr} \left( \frac{dA}{dt}(0) \right)$.

(d) Suppose $A(t)$ is as in part (b), and suppose $\det A(t) \neq 0$ for all $t \in \mathbb{R}$. Show $\frac{d}{dt} \log(\det A(t)) = \sum_{i=1}^n \sum_{j=1}^n (A^{-1})_{ji} \frac{d}{dt}(a_{ij}(t))$, where $(A^{-1})_{ji}$ is the $j^{th}$ entry of $A(t)^{-1}$.

(5) This problem continues and extends problem 2 on Problem Set 1.

(a) (extension of 2(b) on P.S.1) Let $\{a_0, a_1, \ldots, a_n\}$ be distinct real numbers. Show that the linear functionals $f_0, \ldots, f_n$ on $P_n$ defined by $f_k(p) = p(a_k)$ form a basis of $P^*_n$.

(b) Define polynomials $\ell_0, \ldots, \ell_n \in P_n$ by $\ell_k(x) = \prod_{0 \leq i \leq n, i \neq k} \left( \frac{x-a_i}{a_k-a_i} \right)$. Show that $\ell_0, \ldots, \ell_n$ are linearly independent, and thus form a basis of $P_n$.* (Hint: first show $\ell_k(a_i) = \delta_{ik}$.) The $\ell_k$’s are called the Lagrange polynomials (associated with the set $\{a_0, \ldots, a_n\}$).

(c) Show that $\{f_0, \ldots, f_n\}$ in part (a) form the basis of $P^*_n$ which is dual to the basis $\{\ell_0, \ldots, \ell_n\}$ of $P_n$.

(d) Show that if $c_0, \ldots, c_n \in \mathbb{C}$, then $p(x) = \sum_{k=0}^n c_k \ell_k(x)$ is the unique element of $P_n$ satisfying the interpolation conditions $p(a_k) = c_k$, $0 \leq k \leq n$. This result is called the Lagrange interpolation formula. It provides an explicit formula for...
a polynomial $p$ of degree at most $n$ having prescribed values $c_0, \ldots, c_n$ at the prescribed distinct points $a_0, \ldots, a_n$.

(6) Formulate and prove a precise statement of the following fact:
If $\{v_1, \ldots, v_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ (clearly $k \leq n$), and each of $v_1, \ldots, v_k$ is perturbed slightly, then the resulting set of vectors is also linearly independent.