Reading  Horn & Johnson, Chapter 0
Review linear algebra from Halmos, Schaum’s Outline, Hoffman and Kunze, or your choice of similar linear algebra text.

(1) Let \( P_\mathbb{C}^n \) denote the vector space of polynomials of degree \( \leq n \) over the field \( \mathbb{C} \).
   (a) For any \( x_0 \in \mathbb{C} \) show that the polynomials \( (x-x_0)^k, \ k=0,1,\ldots,n \) form a basis for \( P_\mathbb{C}^n \).
   (b) Given \( x_1, x_2 \in \mathbb{C} \) with \( x_1 \neq x_2 \), compute the entries \( a_{st} \) (as functions of \( x_1 \) and \( x_2 \)) for the change of basis matrix \( A \) associated with passing from coordinates in the basis \( (x-x_i)^k, \ k=0,1,\ldots,n \) to coordinates in the basis \( (x-x_j)^k, \ k=0,1,\ldots,n \).
   \( \text{(Hint: Think Taylor series.)} \)
   (c) Given \( x_0 \in \mathbb{C} \), use the differential operator to represent the basis for \( P_\mathbb{C}^n \) that is dual to the basis \( (x-x_0)^k, \ k=0,1,\ldots,n \).

(2) Let \( V = \left\{ u \in C^4(\mathbb{R}) : (\frac{d}{dx})^4 u = (\frac{d}{dx})^2 u \right\} \) and let \( W \subset V \) be the subspace of functions satisfying \( u(0) = 0 \) in addition.
   (a) Find bases for \( V \) and \( W \).
   (b) Show that \( u \mapsto \frac{d}{dx} u \) defines a linear transformation: \( V \to V \). Hence we can restrict the domain to \( W \) to obtain a linear transformation \( L : W \to V \).
   (c) Find the matrix of \( L \) with respect to your bases in part (a).

(3) Let \( P_n \) denote the vector space of polynomials of degree \( \leq n \). For \( 0 \leq k \leq n \), define a linear functional \( f_k \in P^*_n \) by \( f_k(p) = p(k) \) for \( p \in P_n \). Let \( \{e_0, \ldots, e_n\} \) be the basis for \( P^*_n \) dual to the basis \( \{1, x, \ldots, x^n\} \) of \( P_n \).
   (a) Express \( f_k \) as a linear combination of the \( e_i \)'s.
   (b) Show that \( \{f_0, \ldots, f_n\} \) is a basis for \( P^*_n \). \( \text{(Hint: Vandermonde matrix)} \)

(4) Show that if \( 1 \leq p < q \leq \infty \), then \( \ell^p \subseteq \ell^q \).

(5) Let \( [a, b] \subset \mathbb{R} \) be a closed bounded interval, and let \( \Omega_n = \{x_0, \ldots, x_n\} \) be a fixed partition of \( [a, b] \), i.e., \( a = x_0 < x_1 < \cdots < x_n = b \). Let \( m \geq 1 \) be an integer. A function \( s : [a, b] \to \mathbb{R} \) satisfying (i) \( s \in C^{m-1}[a,b] \) and (ii) for \( 0 \leq k \leq n-1 \), \( s \in P_m \) for \( x \in [x_k, x_{k+1}] \) (where \( P_m \) = polynomials of degree \( \leq m \) with real coefficients) is called a (polynomial) \textit{spline} of degree \( m \). Splines are used to approximate general functions. Let \( S_m(\Omega_n) \) be the set of all splines of degree \( m \) with partition \( \Omega_n \).
   (a) Show that \( S_m(\Omega_n) \) is a subspace of \( C^{m-1}[a,b] \).
   (b) For \( k = 0, 1, \ldots, m \), let \( p_k(x) = x^k \). For \( 1 \leq \ell \leq n-1 \), let \( q_{\ell}(x) = [(x-x_\ell)^+]^m \) (where \( (y)_+ = y \) if \( y \geq 0 \), \( (y)_+ = 0 \) if \( y < 0 \)). Show that \( \{p_0, \ldots, p_m, q_1, \ldots, q_{n-1}\} \) is a basis of \( S_m(\Omega_n) \), and conclude that \( \dim(S_m(\Omega_n)) = m+n \).
   (c) Fix \( h > 0 \), let \( n = 4, m = 3 \), and set \( x_k = x_0 + \ell h \) for \( 0 \leq \ell \leq 4 \). Show that there exists a unique \( s \in S_3(\Omega_4) \) satisfying (I) \( (\frac{d}{dx})^k s(x_0) = (\frac{d}{dx})^k s(x_4) = 0 \) for \( k = 0, 1, 2, \) and (II) \( \int_{x_0}^{x_4} s(x)dx = 1 \). \( \text{(s is called a cubic B-spline). Express s in terms of the basis in part (b).} \)

(6) Let \( W_1, W_2 \) be subspaces of a vector space \( V \) and suppose that \( W_2 \) is finite dimensional and \( V = W_1 \oplus W_2 \). Show that
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\dim V = \text{codim} W_1.
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