Math 554 Autumn 2014

Homework Set 1 **Due Friday October 3**

Reading Horn & Johnson, Chapter 0
Review linear algebra from Halmos, Schaum's Outline, Hoffman and Kunze, or your choice of similar linear algebra text.

(1) Let $\mathcal{P}_n(\mathbb{C})$ denote the vector space of polynomials of degree $\leq n$ over the field $\mathbb{C}$.
  (a) For any $x_0 \in \mathbb{C}$ show that the polynomials $(x-x_0)^k,\ k=0,1,\dots,n$ form a basis for $\mathcal{P}_n(\mathbb{C})$.
  (b) Given $x_1, x_2 \in \mathbb{C}$ with $x_1 \neq x_2$, compute the entries a_{st} (as functions of x_1 and x_2) for the change of basis matrix A associated with passing from coordinates in the basis $(x-x_1)^k,\ k=0,1,\dots,n$ to coordinates in the basis $(x-x_2)^k,\ k=0,1,\dots,n$. (*Hint*: Think Taylor series.)
  (c) Given $x_0 \in \mathbb{C}$, use the differential operator to represent the basis for $\mathcal{P}_n(\mathbb{C})^*$ that is dual to the basis $(x-x_0)^k,\ k=0,1,\dots,n$.

(2) Let $V = \left\{u \in C^4(\mathbb{R}) : \left(\frac{d}{dx}\right)^4 u = \left(\frac{d}{dx}\right)^2 u\right\}$ and let $W \subset V$ be the subspace of functions satisfying $u(0)=0$ in addition.
  (a) Find bases for V and W.
  (b) Show that $u \mapsto \frac{d}{dx}u$ defines a linear transformation: $V \to V$. Hence we can restrict the domain to W to obtain a linear transformation $L : W \to V$.
  (c) Find the matrix of L with respect to your bases in part (a).

(3) Let $\mathcal{P}_n$ denote the vector space of polynomials of degree $\leq n$. For $0 \leq k \leq n$, define a linear functional $f_k \in \mathcal{P}_n^*$ by $f_k(p) = p(k)$ for $p \in \mathcal{P}_n$. Let $\{e_0, \dots, e_n\}$ be the basis for $\mathcal{P}_n^*$ dual to the basis $\{1, x, \dots, x^n\}$ of $\mathcal{P}_n$.
  (a) Express f_k as a linear combination of the e_i's.
  (b) Show that $\{f_0, \dots, f_n\}$ is a basis for $\mathcal{P}_n^*$. (Hint: Vandermonde matrix)

(4) Show that if $1 \leq p < q \leq \infty$, then $\ell^p \subsetneq \ell^q$.

(5) Let $[a,b] \subset \mathbb{R}$ be a closed bounded interval, and let $\Omega_n = \{x_0, \dots, x_n\}$ be a fixed partition of $[a,b]$, i.e., $a = x_0 < x_1 < \cdots < x_n = b$. Let $m \geq 1$ be an integer. A function $s : [a,b] \to \mathbb{R}$ satisfying (i) $s \in C^{m-1}[a,b]$ and (ii) for $0 \leq k \leq n-1$, $s \in \mathcal{P}_m$ for $x \in [x_k, x_{k+1}]$ (where $\mathcal{P}_m$ = polynomials of degree $\leq m$ with real coefficients) is called a (polynomial) *spline* of degree m. Splines are used to approximate general functions. Let $S_m(\Omega_n)$ be the set of all splines of degree m with partition Ω_n.
  (a) Show that $S_m(\Omega_n)$ is a subspace of $C^{m-1}([a,b])$.
  (b) For $k = 0,1,\dots,m$, let $p_k(x) = x^k$. For $1 \leq \ell \leq n-1$, let $q_\ell(x) = [(x-x_\ell)_+]^m$ (where $(y)_+ = y$ if $y \geq 0$, $(y)_+ = 0$ if $y < 0$). Show that $\{p_0, \dots, p_m, q_1, \dots, q_{n-1}\}$ is a basis of $S_m(\Omega_n)$, and conclude that $\dim(S_m(\Omega_n)) = m+n$.
  (c) Fix $h > 0$, let $n = 4$, $m = 3$, and set $x_k = x_0 + \ell h$ for $0 \leq \ell \leq 4$. Show that there exists a unique $s \in S_3(\Omega_4)$ satisfying (I) $\left(\frac{d}{dx}\right)^k s(x_0) = \left(\frac{d}{dx}\right)^k s(x_4) = 0$ for $k = 0,1,2$, and (II) $\int_{x_0}^{x_4} s(x)dx = 1$. ($s$ is called a cubic B-spline). Express s in terms of the basis in part (b).

(6) Let W_1, W_2 be subspaces of a vector space V and suppose that W_2 is finite dimensional and $V = W_1 \oplus W_2$. Show that

$$\dim W_2 = \mathrm{codim} W_1.$$