1. In this problem we study the theory and algorithmic implications for an interior point approach to the *horizontal linear complementarity problem*. In what follows we assume that the matrix $Q \in \mathbb{R}^{p \times p}$ is symmetric and positive semi-definite and that $c \in \mathbb{R}^p$, $A \in \mathbb{R}^{m \times p}$, $E \in \mathbb{R}^{k \times p}$, $d \in \mathbb{R}^k$, and $b \in \mathbb{R}^m$.

(a) Assume that Nul $(E^T) = \{0\}$ and consider the QP

$$Q_2 \quad \text{minimize} \quad \frac{1}{2} u^T Q u - c^T u$$
subject to $A u \leq b$, $E u = d$, $0 \leq u$. 

Related to $Q_2$ is the so-called *horizontal LCP*

**The Horizontal LCP (HLCP)**

Given $M \in \mathbb{R}^{m \times n}$, $G \in \mathbb{R}^{k \times n}$, $h \in \mathbb{R}^k$, and $q \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, and $y \in \mathbb{R}^n$ such that

$$M x + G^T z + q = y, \quad G x = h, \quad 0 \leq x, \quad 0 \leq y, \quad \text{and } x^T y = 0.$$

i. Write the KKT conditions for $Q_2$.

ii. Show that the KKT conditions for $Q_2$ are an instance of the HLCP by specifying $M$, $G$, $h$, and $q$ in terms of $Q$, $A$, $E$, $c$, $b$, and $d$.

iii. Show that under this specification Nul $(E^T) = \{0\}$ if and only if Nul $(G^T) = \{0\}$.

iv. Again, show that under this specification the positive semi-definiteness of $Q$ implies that $M$ is positive semi-definite.

(b) Define

$$F(x, z, y) = \begin{bmatrix} M x + G^T z - y + q \\ G x - h \\ X Y e \end{bmatrix}.$$ 

i. Show that $(x, z, y)$ solves the HLCP if and only if $F(x, z, y) = 0$ and $0 \leq x, y$.

ii. Assume that $M$ is positive semi-definite and Nul $(G^T) = \{0\}$. Show that $F(x, z, y)$ is nonsingular whenever $0 < x$ and $0 < y$.

iii. Define

$$\mathcal{F}_+ = \{(x, z, y) : M x + G^T z + q = y, \ G x = h, \ 0 < x, \ 0 < y\}.$$ 

Under the assumption that $M$ is positive semi-definite, Nul $(G^T) = \{0\}$, and $\mathcal{F}_+ \neq \emptyset$, show that the set

$$\mathcal{F}(t) = \{(x, z, y) : M x + G^T z + q = y, \ G x = h, \ 0 \leq x, \ 0 \leq y, \ x^T y \leq t\}$$

is compact for all $t \geq 0$.

(*Hint:*) There are a number of ways to show this. The best way to start is to first show that for all $(x, z, y) \in \mathcal{F}(t)$ $(x, y)$ is bounded in 1-norm. This is done using exactly the same kind of argument as is used in the LCP case. To show that the $z$ component is also bounded there are again a number of possible arguments. The most direct argument uses the fact that the matrix $G G^T$ is invertible (see the midterm exam).
(c) Define
\[ \mathcal{F}_+ = \{ (x, y) : \exists z \in \mathbb{R}^k \text{ such that } (x, z, y) \in \mathcal{F}_+ \}. \]
Assume that \( M \) is positive semi-definite and \( \text{Nul}(G^T) = \{0\} \), and \( \mathcal{F}_+ \neq \emptyset \). Consider the mapping \( u : \mathcal{F}_+ \to \text{int}(\mathbb{R}_+^n) \) given by
\[ u(x, y) = X Ye \]

\[
\begin{align*}
\text{i.} & \quad \text{Show that to each } (x, y) \in \mathcal{F}_+ \text{ there exists a unique } z \in \mathbb{R}^k \text{ such that } (x, z, y) \in \mathcal{F}_+. \\
\text{ii.} & \quad \text{Show that for every } a \in \text{int}(\mathbb{R}_+^n) \text{ there exists a unique } (x, y) \in \mathcal{F}_+ \text{ such that } u(x, y) = a. \quad \text{Using this fact one can show that } u \text{ defines a diffeomorphism between } \mathcal{F}_+ \text{ and } \text{int}(\mathbb{R}_+^n). \\
\text{iii.} & \quad \text{Define a notion of central path for the HLCP and use the previous result to show that it exists under the given hypotheses.} \\
\text{iv.} & \quad \text{Show that a solution the the HLCP exists.}
\end{align*}
\]

2. Program the practical infeasible interior point algorithm described in the notes and test it on the following test problems.

Test Problems:

The matrix \( M \) is computed as follows: Let \( A, B \in \mathbb{R}^{m \times n} \) and \( q, d \in \mathbb{R}^n \) be randomly generated such that \( a_{ij}, b_{ij} \in (-5, 5), q_i \in (-500, 500), d_i \in (0.0, 0.3) \) and that \( B \) is skew-symmetric. Define \( M = A^T A + B + \text{diag}(d) \). Generate ten problems in this way for each of the dimensions \( n = 50, 100, 150, 200 \). Report the maximum, average, and minimum number of iterations needed by the algorithm to solve these problems. Also report back graphs of per-iteration data on a random problem that allow you to discuss the performance of the algorithm. Can you improve the performance of the algorithm by changing some of the parameters appearing in the algorithm?