(1) Suppose \( \Omega = \{ x ; Ax \leq b, Ex = h \} \) where \( A \in \mathbb{R}^{m \times n}, E \in \mathbb{R}^{k \times n}, b \in \mathbb{R}^{m}, \) and \( h \in \mathbb{R}^{k} \).

(a) Given \( x \in \Omega \), show that \( T_{\Omega}(x) = \{ d : A_{i}.d \leq 0 \text{ for } i \in I(x), Ed = 0 \} \), where \( A_{i} \) denotes the \( i \)th row of the matrix \( A \) and \( I(x) = \{ i : A_{i}.x = b_{i} \} \).

(b) Given \( x \in \Omega \), show that every \( d \in T_{\Omega}(x) \) is a feasible direction for \( \Omega \) at \( x \).

(c) Note that parts (a) and (b) above show that \( T_{\Omega}(x) = \bigcup_{\lambda > 0} \lambda (\Omega - x) \) whenever \( \Omega \) is a convex polyhedral set. Why?

(2) Recall that the first-order necessary conditions for optimality condition for the problem \( \min \{ f(x) | x \in x^{0} + S \} \), where \( f : \mathbb{R}^{n} \to \mathbb{R} \) is smooth and \( S \subset \mathbb{R}^{n} \) is a subspace, is

\[
\nabla f(x) \perp S .
\]

Let \( Q \in \mathbb{R}^{n \times n}, c \in \mathbb{R}^{n}, A \in \mathbb{R}^{m \times n}, \) and \( b \in \mathbb{R}^{m} \) and consider the optimization problem

\[
\mathcal{R} = \min \left\{ \frac{1}{2} x^{T}Qx + c^{T}x | Ax \leq b \right\},
\]

where \( t > 0 \) and we define \( \ln(\mu) = -\infty \) if \( \mu \leq 0 \).

(a) Use the optimality condition \((*)\) to show that the optimality conditions for \( \mathcal{R} \) can be written as

\[
(**) \quad \exists w \in \mathbb{R}_{+}^{m} \text{ s.t. } Ax + y = b, \ Qx - A^{T}w + c = 0 \text{ and } \text{diag}(w) \text{ diag}(y) 1 = t1 ,
\]

where \( 1 \) is always the vector of all ones of the appropriate dimension.

(b) Show that if \((x^{k}, y^{k}, w^{k}, t_{k})\) is a sequence of points satisfying \((**)\) with \( t_{k} \downarrow 0 \), then every cluster point of this sequence \((\bar{x}, \bar{y}, \bar{w}, 0)\) is such that \( \bar{x} \) solves \( \min \left\{ \frac{1}{2} x^{T}Qx + c^{T}x | Ax \leq b \right\} \).

(3) Consider the functions

\[
f(x) = \frac{1}{2} x^{T}Qx - c^{T}x
\]

and

\[
f_{t}(x) = \frac{1}{2} x^{T}Qx - c^{T}x + t\phi(x),
\]

where \( t > 0, Q \in \mathbb{R}^{n \times n} \) is positive definite, \( c \in \mathbb{R}^{n} \), and \( \phi : \mathbb{R}^{n} \to \mathbb{R} \cup \{+\infty\} \) is given by

\[
\phi(x) = \begin{cases} 
-\sum_{i=1}^{n} \ln x_{i}, & \text{if } x_{i} > 0, \ i = 1, 2, \ldots, n, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

Remark: Instead of assuming that \( Q \) is positive definite, one could alternatively assume that there does not exist a \( d \in \ker(Q) \) such that \( c^{T}d > 0 \). If you chose, you can proceed with this problem under this alternative hypothesis.

(a) Show that \( \phi \) is a convex function.

(b) Show that both \( f \) and \( f_{t} \) are convex functions.

(c) Show that the solution to the problem \( \min f_{t}(x) \) always exists and is unique.
(d) Let \( \{t_i\} \) be a decreasing sequence of positive real scalars with \( t_i \downarrow 0 \), and let \( x^i \) be the solution to the problem \( \min f_{t_i}(x) \). Show that if the sequence \( \{x^i\} \) has a cluster point \( \bar{x} \), then \( \bar{x} \) must be a solution to the problem \( \min \{f(x) : 0 \leq x\} \).

**Hint:** Use the KKT conditions for the QP \( \min \{f(x) : 0 \leq x\} \).

(4) Let \( Q \in \mathbb{R}^{n \times n} \) be symmetric and positive definite, \( c \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \), and \( A \in \mathbb{R}^{m \times n} \) with \( \text{Nul}(A^T) = \{0\} \).

(a) Show that the matrix \( AQ^{-1}A^T \) is nonsingular.

(b) Show that \( \bar{x} = Q^{-1}A^T(AQ^{-1}A^T)^{-1}b - [I - Q^{-1}A^T(AQ^{-1}A^T)^{-1}A]Q^{-1}c \)
solves the problem

\[
\begin{align*}
& \text{minimize} & & \frac{1}{2} x^T Q x + c^T x \\
& \text{subject to} & & Ax = b
\end{align*}
\]

(5) Let \( Q \in \mathbb{R}^{n \times n} \) be symmetric and positive definite, and let \( c \in \mathbb{R}^n \). Consider the optimization problem

\[
\min_{0 \leq x} \frac{1}{2} x^T Q x + c^T x .
\]

(a) What is the Lagrangian function for this problem?

(b) Show that the Lagrangian dual is the problem

\[
\max_{y \leq c} -\frac{1}{2} y^T Q^{-1} y = -\min_{y \leq c} \frac{1}{2} y^T Q^{-1} y .
\]

(c) Show that if \( \bar{x}, \bar{y} \in \mathbb{R}^n \) satisfy \( \bar{y} = -Q \bar{x} \), then \( \bar{x} \) solves the primal problem if and only if \( \bar{y} \) solves the dual problem, and the optimal values in the primal and dual coincide.

(6) Let \( Q \in \mathbb{R}^{n \times n} \) be symmetric and positive definite. Consider the optimization problem

\[
\mathcal{P} \quad \text{minimize} & & \frac{1}{2} x^T Q x + c^T x \\
& \text{subject to} & & \|x\|_\infty \leq 1 .
\]

(a) Show that this problem is equivalent to the problem

\[
\hat{\mathcal{P}} \quad \text{minimize} & & \frac{1}{2} x^T Q x + c^T x \\
& \text{subject to} & & -e \leq x \leq e ,
\]

where \( e \) is the vector of all ones.

(b) What is the Lagrangian for \( \hat{\mathcal{P}} \)?

(c) Show that the Lagrangian dual for \( \hat{\mathcal{P}} \) is the problem

\[
\mathcal{D} \quad \max -\frac{1}{2} (y - c)^T Q^{-1} (y - c) - \|y\|_1 = -\min \frac{1}{2} (y - c)^T Q^{-1} (y - c) + \|y\|_1 .
\]

This is also the Lagrangian dual for \( \mathcal{P} \).

(d) Show that if \( \bar{x}, \bar{y} \in \mathbb{R}^n \) satisfy \( \bar{y} = Q \bar{x} + c \), then \( \bar{x} \) solves \( \mathcal{P} \) if and only if \( \bar{y} \) solves \( \mathcal{D} \), and the optimal values in \( \mathcal{P} \) and \( \mathcal{D} \) coincide.

(7) Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) with \( m \ll n \). Assume that the system \( Ax = b \) is consistent.

What is the dual to the sparsity inducing optimization problem

\[
\begin{align*}
& \text{minimize} & & \|x\|_1 \\
& \text{subject to} & & Ax = b .
\end{align*}
\]