Part 2
DYNAMICAL SYSTEMS

1 Control of Differential Equations

The equation $\dot{x}(t) = f(t, x(t), u(t))$ can be studied by fixing the control function $u(\cdot)$ and looking at the ordinary differential equation $\dot{x}(t) = g(t, x(t))$ for the function $g(t, x) = f(t, x, u(t))$. The standard elementary theory of ordinary differential equations can't be applied, however, because that would require $g(t, x)$ to be continuous in t. Continuity won't be available if in obtaining g we allow $u(t)$ merely to be *measurable* in t.

First therefore on our agenda is a statement of conditions guaranteeing that an ordinary differential equation under weaker assumptions than continuity in t has a unique solution. It is important for our purposes that the solution exist not just over an unknown small interval $[t_0, t_0 + \varepsilon]$ but over the entire given interval $[t_0, t_1]$. This requires additional assumptions.

Derivative notation. For a vector-valued mapping $h : R^n \to R^m$ with $h(x) = (h_1(x), \dots, h_m(x))$, we will denote by $\nabla h(x)$ the $m \times n$ matrix of first partial derivatives of $h(x)$ with respect to the components of $x = (x_1, \dots, x_n)$. The rows of $\nabla h(x)$ are thus the gradients $\nabla h_i(x)$. This matrix, which is called the *Jacobian* of h at x, can be used to express directional derivatives of h.

By definition, of course, the mapping h is *differentiable* at x if the difference quotient mappings

$$k_s(w) = [h(x + sw) - h(x)]/s \text{ for } s \neq 0$$

converge to a linear mapping $k : R^n \to R^m$ and do so uniformly over all bounded sets of w. (The latter means that for any $\varepsilon > 0$ and any bounded set W, there exists $\delta > 0$ such that $|k_s(w) - k(w)| < \varepsilon$ for all $w \in W$ when $s \in (0, \delta)$.) The limit is then denoted by $h'(x; w)$. *Continuous differentiability* of h means that this limit depends continuously on x. It is equivalent to having $\nabla h(x)$ exist and depend continuously on x, in which case one has $h'(x; w) = \nabla h(x) w$.

For any matrix $A \in R^{m \times n}$, such as $A = \nabla h(x)$, we shall take the norm $|A|$ to be given by

$$|A| = \max_{w \neq 0} |Aw|/|w| = \max_{|w|=1} |Aw|.$$

Lipschitz continuity. In the theory of differential equations and elsewhere, the concept of Lipschitz continuity has a strong role. A mapping $h : R^n \to R^m$ is said to be *Lipschitz continuous* on a set $X_0 \subset R^n$ if there is a constant $\lambda \geq 0$ such that

$$|h(x') - h(x)| \leq \lambda |x' - x| \; for \;\; all \; x, x' \in X_0.$$

An important fact is that h does satisfy this condition relative to a *convex* set X_0 if h is differentiable with $|\nabla h(x)| \leq \lambda$ for all $x \in X_0$.

This follows from the mean value theorem: In setting

$$\theta(s) = w \cdot h((1-s)x + sx')$$

for any $x, x' \in X_0$ and any choice of $w \in R^m$, we get

$$(d\theta/ds)(s) = w \cdot \nabla h((1-s)x + sx')(x' - x) \text{ for } 0 \leq s \leq 1,$$

so that for some $\sigma \in (0,1)$ we have

$$\begin{aligned} w \cdot (h(x') - h(x)) &= \theta(1) - \theta(0) \;\; = \;\; (d\theta/ds)(\sigma) \\ &= w \cdot \nabla h((1-\sigma)x + \sigma x')(x' - x) \end{aligned}$$

and consequently

$$\begin{aligned} |w \cdot (h(x') - h(x))| &\leq |w||\nabla h((1-\sigma)x + \sigma x')||x' - x| \\ &\leq \lambda |w||x' - x|. \end{aligned}$$

This being true for every w, we can in particular take $w = h(x') - h(x)$ and obtain the inequality claimed. (The convexity of X_0 is needed so that the point $(1-s)x + sx'$ will remain in X_0 for $0 \leq s \leq 1$ and give us $|\nabla h((1-s)x + sx')| \leq \lambda$.)

Note that if h is continuously differentiable and the set X_0 is closed and bounded, then the norm $|\nabla h(x)|$ as a continuous function of x is indeed bounded above on X_0 by some λ.

Theorem 2.1 *For the function $f : [t_0, t_1] \times R^n \to R^n$ suppose that*

1. *$f(t,x)$ is measurable in t for fixed x and differentiable in x for fixed t.*

2. *For any bounded set $X_0 \subset R^n$, there is a constant $\lambda \geq 0$ such that $|\nabla_x f(t,x)| \leq \lambda$ a.e. in $t \in [t_0, t_1]$ for every $x \in X_0$.*

3. *There is a constant $\mu \geq 0$ such that $|f(t,x)| \leq \mu(1+|x|)$ a.e. in $t \in [t_0, t_1]$ for every $x \in R^n$.*

Then for each $a \in R^n$ there is a unique trajectory $x(\cdot) \in \mathcal{A}_n^\infty[t_0, t_1]$ such that

$$\dot{x}(t) = f(t, x(t)) \text{ for a.e. } t \in [t_0, t_1], \text{ and } x(t_0) = a.$$

Some observations about the conditions in this theorem can help in applying it. If the first partial derivatives of $f(t,x)$ in x depend continuously on t and x jointly, then $|\nabla f(t,x)|$ depends continuously on (t,x) and is bounded above on $[t_0, t_1] \times X_0$ for any bounded set $X_0 \in R^n$. In this case the second assumption in Theorem 2.1 is indeed satisfied.

If Assumption 2 in Theorem 2.1 holds globally, that is to say not just for bounded X_0 but for $X_0 = R^n$, then $f(t,x)$ is globally Lipschitz continuous in x essentially uniformly in t: There is a constant λ such that for all $x, x' \in R^n$,

$$|f(t,x') - f(t,x)| \leq \lambda |x' - x| \text{ for a.e. } t \in [t_0, t_1].$$

In this case if in addition the function $f(\cdot, \bar{x})$ belongs to $\mathcal{L}_n^\infty[t_0, t_1]$ for at least one $\bar{x} \in R^n$, the third assumption in Theorem 2.1 will be satisfied too.

The proof of Theorem 2.1 will require the following fact about the preservation of measurability under composition. This fact will also be useful to us later.

Proposition 2.2 *Let $h : [t_0, t_1] \times R^n \to R^m$ be such that $h(t,w)$ is measurable in t for fixed w and continuous in w for fixed t. (Note: These are called the* Carathéodory conditions *on the mapping h.) Suppose $v(t) = h(t, w(t))$, where $w(t)$ is measurable in t. Then $v(t)$ is measurable in t.*

Proof. Our argument will use the characterization of measurability discussed in Section 4 of Part 1. Because $w(\cdot)$ is a measurable function, it is the pointwise a.e. limit of a sequence of piecewise constant functions $w^\nu(\cdot)$. Let $v^\nu(t) = h(t, w^\nu(t))$. We have

$$\lim_{\nu \to \infty} h(t, w^\nu(t)) = h(t, w(t)) \text{ a.e. } t \in [t_0, t_1]$$

by the continuity of h in its second argument. Therefore the functions $v^\nu(\cdot)$ converge pointwise a.e. to $v(\cdot)$. If we can demonstrate that $v^\nu(\cdot)$ is measurable for each ν, we will be able to conclude that $v(\cdot)$ is measurable.

Fix ν. Because $w^\nu(\cdot)$ is piecewise constant, the interval $[t_0, t_1]$ can be partitioned into finitely many subintervals I_k^ν on which $w^\nu(\cdot)$ has a constant value w_k^ν. Define $v_k^\nu(t) = h(t, w_k^\nu)$. Then $v_k^\nu(\cdot)$ is a measurable function by the measurability of h in its first argument. We have

$$v^\nu(t) = v_k^\nu(t) \text{ for } t \in I_k^\nu,$$

so $v^\nu(\cdot)$ is indeed measurable. □

The following growth property is very useful in proving Theorems 2.1 and 2.5.

Proposition 2.3 *Let $x(\cdot) \in \mathcal{A}_n^\infty[t_0, t_1]$ satisfy the estimates*

$$|\dot{x}(t)| \leq \mu_0(t) + \mu_1(t)|x(t)| \text{ for a.e. } t \in [t_0, t_1] \text{ and } |x(t_0)| \leq \alpha,$$

for certain functions $\mu_0(\cdot)$ and $\mu_1(\cdot)$ in $\mathcal{L}_1^\infty[t_0, t_1]$ and a constant $\alpha \geq 0$. Then

$$|x(t)| \leq \rho(t) \text{ for all } t \in [t_0, t_1],$$

where $\rho(\cdot)$ is the unique solution to the linear differential equation

$$\dot{\rho}(t) = \mu_0(t) + \mu_1(t)\rho(t) \text{ for a.e. } t \in [t_0, t_1], \quad \rho(t_0) = \alpha,$$

namely

$$\rho(t) = (\alpha + \bar{\mu}_0(t))e^{\bar{\mu}_1(t)}$$

with $\bar{\mu}_1(t) = \int_{t_0}^t \mu_1(s)ds$ and $\bar{\mu}_0(t) = \int_{t_0}^t e^{-\bar{\mu}_1(s)}\mu_0(s)ds$.

Proof. Let $\xi(t) = |x(t)|$. The function $\xi(\cdot)$ is Lipschitz continuous on $[t_0, t_1]$, because it is composed of the Lipschitz continuous function $x(\cdot)$ and the euclidean norm $|\cdot|$ (which is Lipschitz continuous with modulus 1: $||b| - |a|| \leq |b - a|$). Thus $\xi(\cdot) \in \mathcal{A}_1^\infty[t_0, t_1]$. In particular $\dot{\xi}(t)$ exists for almost every $t \in [t_0, t_1]$. On the open subset of $[t_0, t_1]$ where $\xi(t) > 0$, it is clear from the ordinary chain rule that $\dot{\xi}(t) = \dot{x}(t) \cdot x(t)/|x(t)|$ wherever $\dot{x}(t)$ exists (which is a.e.). At points t strictly between t_0 and t_1 where $\xi(t) = 0$, we must have $\dot{\xi}(t) = 0$ if $\dot{\xi}(t)$ exists at all, because these are points at which the function $\xi(\cdot)$ achieves its minimum over $[t_0, t_1]$. From these facts it follows that $\dot{\xi}(t) \leq |\dot{x}(t)|$ for a.e. $t \in [t_0, t_1]$. Therefore

$$\dot{\xi}(t) \leq \mu_0(t) + \mu_1(t)\xi(t) \text{ a.e., and } \xi(t_0) \leq \alpha.$$

Consider now the function $\theta(t) = \xi(t) - \rho(t)$. This has

$$\theta(t_0) = \xi(t_0) - \rho(t_0) \leq \alpha - \alpha \leq 0$$

and also, for a.e. $t \in [t_0, t_1]$,

$$\dot{\theta}(t) = \dot{\xi}(t) - \dot{\rho}(t) \leq [\mu_0(t) + \mu_1(t)\xi(t)] - [\mu_0(t) + \mu_1(t)\rho(t)] = \mu_1(t)\theta(t).$$

Then the function $\varphi(t) = \theta(t)e^{-\bar{\mu}_1(t)}$ has

$$\dot{\varphi}(t) = (\dot{\theta}(t) - \mu_1(t)\theta(t))e^{-\bar{\mu}_1(t)} \leq 0 \text{ a.e. } t \in [t_0, t_1]$$

and $\varphi(t_0) = \theta(t_0) \leq 0$, so $\varphi(t)$ is nonincreasing in t and in particular satisfies $\varphi(t) \leq 0$ for all $t \in [t_0, t_1]$. This implies that $\theta(t) \leq 0$ for all t, i.e. that $\xi(t) \leq \rho(t)$ for all t, our desired conclusion. □

Proof of Theorem 2.1. For any function $x(\cdot) \in \mathcal{C}_n[t_0, t_1]$, let $M(x(\cdot))$ denote the function $y(\cdot)$ given by

$$y(t) = a + \int_{t_0}^{t} f(s, x(s))ds.$$

Our aim is to show the existence of a unique $x(\cdot)$ such that $M(x(\cdot)) = x(\cdot)$, i.e. a unique *fixed point* of the mapping M. This will be the desired solution to our differential equation.

Before proceeding, we must verify that the function $y(\cdot) = M(x(\cdot))$ is well defined and, like $x(\cdot)$, belongs to $\mathcal{C}_n[t_0, t_1]$, in fact to $\mathcal{A}_n^\infty[t_0, t_1]$. By Proposition 2.2, $f(t, x(t))$ is measurable in t, inasmuch as $x(t)$ is measurable (actually continuous) in t. Furthermore, by Assumption 3 in the theorem the function $v(t) = f(t, x(t))$ has

$$|v(t)| \leq \mu(1 + |x(t)|) \text{ for a.e. } t \in [t_0, t_1],$$

where the function on the right is continuous in t and therefore is bounded above on $[t_0, t_1]$. Thus $v(\cdot)$ belongs to $\mathcal{L}_n^\infty[t_0, t_1]$ and it follows that $y(\cdot)$ is well defined and belongs to $\mathcal{A}_n^\infty[t_0, t_1]$, as claimed.

We show next that the mapping M has *at most one* fixed point, i.e. that the differential equation can't have two different solutions starting at a. Suppose $x(\cdot)$ and $x'(\cdot)$ both were solutions and let r_0 be the larger of the two norms $||x(\cdot)||_\infty$ and $||x'(\cdot)||_\infty$, so that $x(t)$ and $x'(t)$ belong for every $t \in [t_0, t_1]$ to the closed ball of radius r_0 around the origin of R^n. This ball is a bounded convex set. We can apply Assumption 2 of the theorem to this set as X_0 and invoke the property of Lipschitz continuity cited before the statement of the theorem. This will give us a value λ_0 such that

$$|f(t,x'(t)) - f(t,x(t))| \leq \lambda_0 |x'(t) - x(t)| \text{ for a.e. } t \in [t_0, t_1].$$

Consider the function $z(t) = x'(t) - x(t)$. It too belongs to $\mathcal{A}_n^\infty[t_0, t_1]$ and satisfies $\dot{z}(t) = f(t,x'(t)) - f(t,x(t))$ a.e., therefore $|\dot{z}(t)| \leq \lambda_0 |z(t)|$ a.e. By Proposition 2.3 (as applied to $\mu_0(t) \equiv 0$, $\mu_1(t) \equiv \lambda_0$, and $\alpha = |z(t0)|$) we then have $|z(t)| \leq \rho_0(t)$ for all $t \in [t_0, t_1]$, where $\rho_0(t) = |z(t_0)|e^{(t-t_0)\lambda_0}$. But $z(t_0) = x'(t_0) - x(t_0) = a - a = 0$. Thus $z(t) \equiv 0$, i.e. $x'(t) \equiv x(t)$. This finishes the verification of uniqueness.

The rest of the proof aims at establishing existence in a constructive manner. Any fixed point $x(\cdot)$ of M would in particular have

$$|\dot{x}(t)| \leq |f(t,x(t))| \leq \mu(1 + |x(t)|) \text{ a.e. and } |x(t_0)| = |a|$$

by virtue of Assumption 3 of the theorem. By Proposition 2.3 (as applied to $\mu_0(t) = \mu_1(t) \equiv \mu$ and $\alpha = a$) it therefore would have to satisfy $|x(t)| \leq \rho(t)$ for all $t \in [t_0, t_1]$, where

$$\rho(t) = (1 + |a|)e^{(t-t_0)\mu} - 1.$$

Let S be the subset of $\mathcal{C}_n[t_0, t_1]$ consisting of the functions $x(\cdot)$ that satisfy $|x(t)| \leq \rho(t)$ for all t. Obviously every such function has

$$||x(\cdot)||_\infty \leq \rho(t_1).$$

We claim that $M(x(\cdot)) \in S$ for every $x(\cdot) \in S$. Indeed, if $y(\cdot) = M(x(\cdot))$ with $x(\cdot) \in S$, we have $|y(t_0)| = \rho(t_0)$ and

$$|\dot{y}| = |f(t,x(t))| \leq \mu(1 + |x(t)|) \leq \mu(1 + \rho(t)) = \dot{\rho}(t)$$

for a.e. $t \in [t_0, t_1]$ so that $|y(t)| \leq \rho(t)$ for all $t \in [t_0, t_1]$, i.e. $y(\cdot) \in S$.

By Assumption 2 in the theorem, as applied earlier but this time to X_0 equal to the closed ball of radius $\rho(t_1)$ around the origin of R^n, there exists a number $\lambda \geq 0$ such that

$$|f(t,x') - f(t,x)| \leq \lambda |x' - x| \text{ for a.e. } t \in [t_0, t_1]$$

when $|x| \leq \rho(t_1)$ and $|x'| \leq \rho(t_1)$ in R^n. Then

$$|f(t,x'(t)) - f(t,x(t))| \leq \lambda |x'(t) - x(t)| \text{ a.e. for } x(\cdot), x'(\cdot) \in S.$$

In particular, for the functions $y(\cdot) = M(x(\cdot))$ and $y'(\cdot) = M(x'(\cdot))$, which satisfy

$$\begin{aligned}
|y'(t) - y(t)| &= |\int_{t_0}^{t} [f(s, x'(s)) - f(s, x(s))] ds| \\
&\leq \int_{t_0}^{t} |f(s, x'(s)) - f(s, x(s))| ds \\
&\leq \lambda \int_{t_0}^{t} |x'(s) - x(s)| ds \quad \leq \quad (t_1 - t_0)\lambda ||x'(\cdot) - x(\cdot)||_\infty,
\end{aligned}$$

we get $||y'(\cdot) - y(\cdot)||_\infty \leq (t_1 - t_0)\lambda ||x'(\cdot) - x(\cdot)||_\infty$. In other words, the mapping $M : S \to S$ is Lipschitz continuous with modulus $(t_1 - t_0)\lambda$.

Starting now from any $x^0(\cdot) \in S$, let us generate a sequence of functions by $x^\nu(\cdot) = M(x^{\nu-1})$. These functions all belong to S. We shall show they converge to a fixed point $x(\cdot)$ of M, and this will complete the proof.

It will be enough to demonstrate convergence of x^ν to such an $x(\cdot)$ with respect to the norm $||\cdot||_\infty$, but it is valuable to note that the convergence will then necessarily occur also with respect to the stronger norm $|||\cdot|||_\infty$ of the space $\mathcal{A}_n^\infty[t_0, t_1]$. This is because the Lipschitz bound

$$|f(t, x^\nu(t)) - f(t, x(t))| \leq \lambda |x^\nu(t) - x(t)|,$$

when taken together with the equations

$$\dot{x}^{\nu+1}(t) = f(t, x^\nu(t)) \text{ and } \dot{x}(t) = f(t, x(t)) \text{ for a.e. } [t_0, t_1],$$

which correspond to $x^{\nu+1}(\cdot) = M(x^\nu(\cdot))$ and $x(\cdot) = M(x(\cdot))$, implies that

$$|\dot{x}^{\nu+1}(t) - \dot{x}(t)| \leq \lambda |x^\nu(t) - x(t)| \text{ for a.e. } [t_0, t_1].$$

Inasmuch as $x^{\nu+1}(t_0) = x(t_0) = a$, we have by definition

$$\begin{aligned}
|||x^{\nu+1}(\cdot) - x(\cdot)|||_\infty &= \max\{|x^{\nu+1}(t_0) - x(t_0)|, ||\dot{x}^{\nu+1}(\cdot) - \dot{x}(\cdot)||_\infty\} \\
&= ||\dot{x}^{\nu+1}(\cdot) - \dot{x}(\cdot)||_\infty
\end{aligned}$$

and therefore

$$|||x^{\nu+1}(\cdot) - x(\cdot)|||_\infty \leq \lambda ||x^\nu(\cdot) - x(\cdot)||_\infty.$$

Thus if $||x^\nu(\cdot) - x(\cdot)||_\infty \to 0$ we automatically also have $|||x^\nu(\cdot) - x(\cdot)|||_\infty \to 0$

We start now on the proof of the claimed convergence. Divide the interval $[t_0, t_1]$ into $t_0 = s_0 < s_1 < s_2 < \ldots < s_m = t_1$, where the subintervals $I_k = [s_{k-1}, s_k]$ all have length $s_k - s_{k-1} \leq 1/(1+\lambda)$. Observe that over each such subinterval we have

$$\begin{aligned}
&|x^{\nu+1}(t) - x^{\nu}(t)| \\
&\quad = |[x^{\nu+1}(s_{k-1}) - x^{\nu}(s_{k-1})] + \int_{s_{k-1}}^{t} [f(s, x^{\nu}(s)) - f(s, x^{\nu-1}(s))]ds| \\
&\quad \leq |x^{\nu+1}(s_{k-1}) - x^{\nu}(s_{k-1})| + \int_{s_{k-1}}^{t} |f(s, x^{\nu}(s)) - f(s, x^{\nu-1}(s))|ds \\
&\quad \leq |x^{\nu+1}(s_{k-1}) - x^{\nu}(s_{k-1})| + \lambda \int_{s_{k-1}}^{t} |x^{\nu}(s)) - x^{\nu-1}(s)|ds \\
&\quad \leq |x^{\nu+1}(s_{k-1}) - x^{\nu}(s_{k-1})| + (s_k - s_{k-1})\lambda \max_{s \in I_k} |x^{\nu}(s) - x^{\nu-1}(s)|,
\end{aligned}$$

where $(s_k - s_{k-1})\lambda \leq \lambda/(1+\lambda)$. Thus if we define $\beta = \lambda/(1+\lambda)$ and set

$$||x^{\nu+1}(\cdot) - x^{\nu}(\cdot)||_{\infty}^{(k)} = \max_{s \in I_k} |x^{\nu}(s) - x^{\nu-1}(s)|,$$

with the notational interpretation that $||x^{\nu+1}(\cdot) - x^{\nu}(\cdot)||_{\infty}^{(0)} = 0$, we obtain the relations

$$||x^{\nu+1}(\cdot) - x^{\nu}(\cdot)||_{\infty}^{(k)} \leq ||x^{\nu+1}(\cdot) - x^{\nu}(\cdot)||_{\infty}^{(k-1)} + \beta ||x^{\nu}(\cdot) - x^{\nu-1}(\cdot)||_{\infty}^{(k)}$$

for $k = 1, 2, \ldots$, where $\beta < 1$.

¿From these relations we have for $k = 1$ that

$$||x^{\nu+1}(\cdot) - x^{\nu}(\cdot)||_{\infty}^{(1)} \leq \beta ||x^{\nu}(\cdot) - x^{\nu-1}(\cdot)||_{\infty}^{(1)}$$

and therefore by induction that

$$||x^{\nu+1}(\cdot) - x^{\nu}(\cdot)||_{\infty}^{(1)} \leq \beta^{\nu} ||x^{1}(\cdot) - x^{0}(\cdot)||_{\infty}^{(1)}.$$

This implies that

$$\Sigma_{\nu=1}^{\infty} ||x^{\nu}(\cdot) - x^{\nu-1}(\cdot)||_{\infty}^{(1)} < \infty$$

and in particular that in the restricted space $\mathcal{C}_n(I_1)$ the sequence of functions $x^{\nu}(\cdot)$ is a Cauchy sequence, hence converges there to a certain function that we may denote by $x_1(\cdot)$.

Now in general, our relations among the restricted norms give us

$$\begin{aligned}
\Sigma_{\nu=1}^{N} ||x^{\nu+1}(\cdot) - x^{\nu}(\cdot)||_{\infty}^{(k)} \quad \leq \quad & \Sigma_{\nu=1}^{N} ||x^{\nu}(\cdot) - x^{\nu-1}(\cdot)||_{\infty}^{(k-1)} \\
& + \beta \Sigma_{\nu=1}^{N} ||x^{\nu}(\cdot) - x^{\nu-1}(\cdot)||_{\infty}^{(k)}.
\end{aligned}$$

This can be written as

$$\begin{aligned}||x^{N+1}(\cdot) - x^N(\cdot)||_\infty^{(k)} + (1-\beta)\Sigma_{\nu=1}^{N-1}||x^{\nu+1}(\cdot) - x^\nu(\cdot)||_\infty^{(k)} \\ \leq \Sigma_{\nu=1}^{N}||x^{\nu+1}(\cdot) - x^\nu(\cdot)||_\infty^{(k-1)} + \beta||x^1(\cdot) - x^0(\cdot)||_\infty^{(k)},\end{aligned}$$

where $1-\beta > 0$. Thus

$$\begin{aligned}&\Sigma_{\nu=1}^{N}||x^{\nu+1}(\cdot) - x^\nu(\cdot)||_\infty^{(k)} \\ &\quad \leq (1-\beta)^{-1}[\ \Sigma_{\nu=1}^{N}||x^{\nu+1}(\cdot) - x^\nu(\cdot)||_\infty^{(k-1)} + \beta||x^1(\cdot) - x^0(\cdot)||_\infty^{(k)}\ ]\end{aligned}$$

for all N. It follows then that

$$\Sigma_{\nu=1}^{\infty}||x^\nu(\cdot) - x^{\nu-1}(\cdot)||_\infty^{(k-1)} < \infty \quad \Rightarrow \quad \Sigma_{\nu=1}^{\infty}||x^\nu(\cdot) - x^{\nu-1}(\cdot)||_\infty^{(k)} < \infty.$$

By induction, therefore, the sequence of functions x^ν is Cauchy in every one of the restricted spaces $\mathcal{C}_n(I_k)$, $k = 1, 2, \ldots m$, and it converges in them to functions $x_k(\cdot)$. The latter agree at the join points and constitute a single function $x(\cdot)$ in $\mathcal{C}_n[t_0, t_1]$. In each $\mathcal{C}_n(I_k)$ we have

$$\lim_{\nu\to\infty} ||x^\nu(\cdot) - x(\cdot)||_\infty^{(k)} = 0,$$

and since

$$||x^\nu(\cdot) - x(\cdot)||_\infty = \max_{k=1,\ldots,m} ||x^\nu(\cdot) - x(\cdot)||_\infty^{(k)},$$

this gives us

$$\lim_{\nu\to\infty} ||x^\nu(\cdot) - x(\cdot)||_\infty = 0.$$

In other words, $x^\nu(\cdot)$ converges to $x(\cdot)$ in $\mathcal{C}_n[t_0, t_1]$. Since $x^{\nu+1}(\cdot) = M(x^\nu(\cdot))$ and M is a continuous mapping on S (actually Lipschitz continuous), we have in the limit that $x(\cdot) = M(x(\cdot))$. Thus $x(\cdot)$ is a fixed point of M, and our proof of Theorem 2.1 is complete. □

Remarks. The proof of Theorem 2.1 has established some important facts beyond the statement of the theorem itself. It has shown for instance that the uniqueness of the solution holds relative to the larger space $\mathcal{A}_n^1[t_0, t_1]$: if $x(\cdot)$ in this space satisfies the differential equation, then $x(\cdot)$ actually must be in $\mathcal{A}_n^\infty[t_0, t_1]$. More noteworthy perhaps is the *constructive* nature of the proof. We can start with any function $x^0(\cdot)$ in $C_n[t_0, t_1]$ having $x^0(t_0) = a$, for instance the constant function $x^0(t) \equiv a$, and by defining

$$x^{\nu+1}(t) = a + \int_{t_0}^{t} f(s, x^\nu(s)ds \text{ for } \nu = 0, 1, \ldots$$

get a sequence of functions in $\mathcal{A}_n^\infty[t_0, t_1]$ that converges in the norm of $\mathcal{A}_n^\infty[t_0, t_1]$ to the desired solution $x(\cdot)$.

Example. The case of $f(t,x) = A(t)x + b(t)$ is that of a *linear* differential equation. It satisfies all the assumptions of Theorem 2.1 as long as the components of $A(t)$ and $b(t)$ are measurable, essentially bounded functions of $t \in [t_0, t_1]$. One can take $\lambda = ||A(\cdot)||_\infty$ and $\mu = \max\{\lambda, ||b(\cdot)||_\infty\}$.

Example. The need for Assumption 3 in Theorem 2.1, in order to obtain a solution $x(\cdot)$ over the whole interval $[t_0, t_1]$ instead of just some unknown subinterval $[t_0, t_0 + \varepsilon]$, is shown by the classical equation $\dot{x}(t) = 1 + x(t)^2$, $x(0) = 0$, on the interval $[0, T]$ for any $t \geq \pi/2$. The unique solution over any interval $[0, \tau]$ with $\tau < \pi/2$ is $x(t) = \arctan t$, and this function tends to ∞ as t approaches $\pi/2$. There is no hope then of having a solution over the entire interval $[0, T]$. The critical value $t = \pi/2$ is called the *escape time* for this equation. Assumption 3 makes sure that no such escape time is encountered in the interval $[t_0, t_1]$.

We are able now to answer questions about control systems by applying Theorem 2.1 to a composite function.

Theorem 2.4 *Let $f : [t_0, t_1] \times R^n \times R^d \to R^n$ be such that*

1. *$f(t,x,u)$ is measurable in t for fixed x and u, continuous in (x,u) for fixed t, and differentiable in x for fixed t and u.*

2. *For any bounded sets $X_0 \subset R^n$ and $U_0 \subset R^d$ there is a constant $\lambda \geq 0$ such that $|\nabla_x f(t,x,u)| \leq \lambda$ a.e. in $t \in [t_0, t_1]$ for every $x \in X_0$ and $u \in U_0$.*

3. *For any bounded set $U_0 \subset R^d$ there is a constant μ such that $|f(t,x,u)| \leq \mu(1 + |x|)$ for a.e. $t \in [t_0, t_1]$ and every $x \in R^n$ when $u \in U_0$.*

Then for each $a \in R^n$ and $u(\cdot) \in \mathcal{L}_d^\infty[t_0, t_1]$ there is a unique trajectory $x(\cdot) \in \mathcal{A}_n^\infty[t_0, t_1]$ satisfying

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ for a.e. } t \in [t_0, t_1] \text{ and } x(t_0) = a.$$

Proof. Fix any $\bar{u}(\cdot) \in \mathcal{L}_d^\infty[t_0, t_1]$ and define $\bar{f}(t,x) = f(t, x, \bar{u}(t))$. To obtain our result, we need only demonstrate that $\bar{f}$ satisfies the hypothesis of Theorem 2.1.

Clearly $\bar{f}(t,x)$ is differentiable in x for each fixed t by Assumption 1 on f. This assumption also gives us $f(t,x,\bar{u}(t))$ measurable in t for fixed x by Proposition 2.2. Thus $\bar{f}$ satisfies Assumption 1 in Theorem 2.1.

Let $\bar{r} = ||\bar{u}(\cdot)||_\infty$ and take U_0 to be the set of all $u \in R^d$ having $|u| \leq \bar{r}$. Then Assumptions 2 and 3 of the present theorem yield Assumptions 2 and 3 of Theorem 2.1. This is all that had to be verified. □

A particular case where Assumptions 1 and 2 of Theorem 2.4 are fulfilled is the common one where $f(t,x,u)$ and $\nabla_x f(t,x,u)$ depend continuously on t, x and u jointly, and there is a continuous function $\mu_0 : R^d \to R$ such that

$$|f(t,x,u)| \leq \mu_0(u)(1+|x|) \text{ for all } t \in [t_0,t_1], x \in R^n \text{ and } u \in R^d.$$

Example. The linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = a$$

meets the assumptions in Theorem 2.4 if the components of the matrices $A(t)$ and $B(t)$ are measurable, essentially bounded functions of $t \in [t_0,t_1]$. This is true in particular, of course, if these components are continuous or even constant functions of t.

2 Approximate Controls and Linearization

The concept of control can be seen now in terms of a mapping $\mathcal{F}$ that assigns to each choice of initial state a and control function $u(\cdot)$ in $\mathcal{L}_d^\infty[t_0,t_1]$ a uniquely determined trajectory $x(\cdot)$ in $\mathcal{A}_n^\infty[t_0,t_1]$, denoted by $x(\cdot) = \mathcal{F}(a,u(\cdot))$. Theorem 2.4 sets down general conditions on the control equation

$$\dot{x}(t) = f(t,x(t),u(t)) \text{ a.e. } t \in [t_0,t_1], \quad x(t_0) = a,$$

under which this mapping $\mathcal{F} : R^n \times \mathcal{L}_d^\infty[t_0,t_1] \to \mathcal{A}_n^\infty[t_0,t_1]$ is well defined. Many questions can then be asked about properties of $\mathcal{F}$ that are of interest in their own right and relevant also to a later investigation of controllability and *optimal* control.

The continuity of $\mathcal{F}$ is obviously an important issue. Suppose $x^\nu(\cdot) = \mathcal{F}(a^\nu,u^\nu(\cdot))$, where $a^\nu \to a$ in R^n and $u^\nu(\cdot) \to u(\cdot)$ in $\mathcal{L}_d^\infty[t_0,t_1]$, that is to say, $||u^\nu(\cdot) - u(\cdot)||_\infty \to 0$. Do the functions $x^\nu(\cdot)$ then converge to the function $x(\cdot) = \mathcal{F}(a,u(\cdot))$ in $\mathcal{A}_n^\infty[t_0,t_1]$ in the sense that $|||x^\nu(\cdot) - x(\cdot)|||_\infty \to$

0? This would mean that $\mathcal{F}$ is continuous as a mapping from the Banach space $R^n \times \mathcal{L}_d^\infty[t_0, t_1]$ (whose norm can be taken to be $||(a, u(\cdot))|| = \max\{|a|, ||u(\cdot)||_\infty\}$, for instance) to the Banach space $\mathcal{A}_n^\infty[t_0, t_1]$.

The question here is really one of approximation. Suppose we want to generate the trajectory $x(\cdot) = \mathcal{F}(a, u(\cdot))$ but for one reason or another are unable to implement the control function $u(\cdot)$. Perhaps $u(\cdot)$ is discontinuous, and the mechanism we are modelling can only deal in practice with continuous controls. We have to consider what will happen when we replace $u(\cdot)$ by an implementable approximating control $u^\nu(\cdot)$, where the approximation gets better and better as $\nu \to \infty$. Another consideration might be that we are somewhat uncertain of the true initial state a but can approximate it by a^ν, where $a^\nu \to a$. The trajectory for $u^\nu(\cdot)$ and a^ν is $x^\nu(\cdot)$. Will this get closer and closer to $x(\cdot)$ as our approximations improve?

We have posed the continuity property of $\mathcal{F}$ in terms of a particular kind of approximation for control functions, namely the one dictated by the norm in $\mathcal{L}_d^\infty[t_0, t_1]$. For most purposes, however, this kind of approximation demands too much. We cannot, for instance, in this sense approximate a *discontinuous* control function $u(\cdot)$ by a sequence of *continuous* control functions $u^\nu(\cdot)$, because the limit of a sequence of continuous functions that converges with respect to the norm $||\cdot||_\infty$ is necessarily another continuous function.

A natural property to consider instead is that of convergence of $u^\nu(\cdot)$ pointwise a.e. to $u(\cdot)$. Indeed, we know that any function $u(\cdot)$ in the space $\mathcal{L}_d^\infty[t_0, t_1]$ can be approximated in this sense by a sequence of functions $u^\nu(\cdot)$ that are continuous, or for that matter piecewise constant. This kind of approximation can be effected even with $||u^\nu(\cdot)||_\infty \le ||u(\cdot)||_\infty$ for all ν, if desired.

The next theorem shows that $\mathcal{F}$ behaves continuously under this broader kind of approximation, provided that we also appropriately weaken the demands made on the convergence of $x^\nu(\cdot)$ to $x(\cdot)$. It is no longer appropriate to expect $|||x^\nu(\cdot) - x(\cdot)|||_\infty \to 0$, which would entail $||\dot{x}^\nu(\cdot) - \dot{x}(\cdot)||_\infty \to 0$. Rather we need to look for pointwise a.e. convergence of these derivatives.

Norm convergence of $u^\nu(\cdot)$ to $u(\cdot)$ in $\mathcal{L}_d^\infty[t_0, t_1]$ does, of course, in particular imply pointwise convergence almost everywhere. Later, when we come to discuss the differentiability of $\mathcal{F}$, we shall return to the question of the continuity of $\mathcal{F}$ with respect to the norms on $\mathcal{L}_d^\infty[t_0, t_1]$ and $\mathcal{A}_n^\infty[t_0, t_1]$.

Theorem 2.5 *For a control system*

$$\dot{x}(t) = f(t, x(t), u(t)) \ \ for\ a.e.\ t \in [t_0, t_1] \quad and \quad x(t_0) = a$$

in which f satisfies the conditions in Theorem 2.4, the associated mapping $\mathcal{F} : (a, u(\cdot)) \mapsto x(\cdot)$ has the following property. If the initial states a^ν in R^n converge to a and the control functions $u^\nu(\cdot)$ in $\mathcal{L}_d^\infty[t_0, t_1]$ converge pointwise a.e. on $[t_0, t_1]$ to $u(\cdot)$ with the norms $||u^\nu(\cdot)||_\infty$ all bounded by some number r, then the corresponding trajectories $x^\nu(\cdot) = \mathcal{F}(a^\nu, u^\nu(\cdot))$ converge to $x(\cdot) = \mathcal{F}(a, u(\cdot))$ with respect to the norm $||\cdot||_\infty$ as elements of $\mathcal{C}_n[t_0, t_1]$, while their derivatives $\dot{x}^\nu(\cdot)$ converge pointwise a.e. to $\dot{x}(\cdot)$ on $[t_0, t_1]$.

Proof. Let α_0 be an upper bound for the sequence of norms $|a^\nu|$. Let U_0 be the closed ball of radius r around the origin of R^d, and let μ be the corresponding growth constant for f whose existence is guaranteed by Assumption 3 in Theorem 2.4. We have $u^\nu(t) \in U_0$ a.e., so

$$|\dot{x}^\nu(t)| = |f(t, x^\nu(t), u^\nu(t))| \leq \mu(1 + |x^\nu(t)|) \text{ for a.e. } t \in [t_0, t_1].$$

Applying Proposition 2.3 (with $\mu_0(t) = \mu_1(t) \equiv \mu$ and $\alpha = \alpha_0$), we see that

$$|x^\nu(t)| \leq \rho(t) = (\alpha_0 + 1)e^{(t-t_0)\mu} - 1 \quad \text{for all } t \in [t_0, t_1].$$

This holds for the trajectory $x(\cdot) = \mathcal{F}(a, u(\cdot))$ too, by the same reasoning.

Let X_0 be the closed ball of radius $\rho(t_1)$ around the origin of R^n. Then $x^\nu(t)$ and $x(t)$ belong to X_0 for all $t \in [t_0, t_1]$. Take λ to be the constant corresponding to f, X_0, and U_0 in Assumption 2 of Theorem 2.4. This bound yields the Lipschitz estimate

$$|f(t, x^\nu(t), w) - f(t, x(t), w)| \leq \lambda |x^\nu(t) - x(t)| \text{ a.e. } t \text{ for any } w \in U_0.$$

Consider now the functions $y^\nu(t) = x^\nu(t) - x(t)$. These satisfy

$$\begin{aligned} |\dot{y}^\nu(t)| &= |x^\nu(t) - x(t)| \ = \ |f(t, x^\nu(t), u^\nu(t)) - f(t, x(t), u(t))| \\ &\leq |f(t, x^\nu(t), u^\nu(t)) - f(t, x(t), u^\nu(t))| \\ &\qquad\qquad + |f(t, x(t), u^\nu(t)) - f(t, x(t), u(t))| \\ &\leq \lambda |x^\nu(t) - x(t)| + \mu_0^\nu(t) = \lambda |y^\nu(t)| + \mu_0^\nu(t) \end{aligned}$$

for almost every $t \in [t_0, t_1]$, where the functions

$$\mu_0^\nu(t) = |f(t, x(t), u^\nu(t)) - f(t, x(t), u(t))|$$

are measurable (by Proposition 2.2 and Assumption 1 of Theorem 2.4 on f) with

$$\begin{aligned}\mu_0^\nu(t) &\leq |f(t,x(t),u^\nu(t))| + |f(t,x(t),u(t))| \\ &\leq 2\mu(1+|x(t)|) \leq 2\mu(1+\rho(t_1))\end{aligned}$$

and satisfy

$$\mu_0^\nu(t) \to 0 \text{ a.e. } t \in [t_0,t_1] \text{ as } \nu \to \infty.$$

Applying Proposition 2.3 (with $\mu_0(t) = \mu_0^\nu(t)$, $\mu_1(t) \equiv \lambda$, and $\alpha = |y^\nu(t_0)| = |a^\nu - a|$), we obtain

$$|y^\nu(t)| \leq (|a^\nu - a| + \int_{t_0}^{t} e^{-(s-t_0)\lambda}\mu_0^\nu(s)ds)e^{(t-t_0)\lambda} \text{ for all } t \in [t_0,t_1]$$

and consequently

$$||y^\nu(\cdot)||_\infty \leq [|a^\nu - a| + \int_{t_0}^{t_1} e^{-(s-t_0)\lambda}\mu_0^\nu(s)ds]e^{(t_1-t_0)\lambda}.$$

The properties cited for the functions $\mu_0^\nu(\cdot)$ imply that the integrals $\int_{t_0}^{t_1} e^{-(s-t_0)\lambda}\mu_0^\nu(s)ds$ converge to 0 as $\nu \to \infty$. From this and the Assumption that $|a^\nu - a| \to 0$ we may conclude that $||y^\nu(\cdot)||_\infty \to 0$. But $y^\nu(\cdot) = x^\nu(\cdot) - x(\cdot)$. Therefore the functions $x^\nu(\cdot)$ converge to $x(\cdot)$ as elements of $\mathcal{C}_n[t_0,t_1]$, as claimed.

At the same time we see from the inequality

$$|\dot{y}^\nu(t)| \leq \lambda|y^\nu(t)| + \mu_0^\nu(t)$$

and the limit $||y^\nu(\cdot)||_\infty \to 0$ that $|\dot{y}^\nu(t)| \to 0$ for a.e. $t \in [t_0,t_1]$. Therefore $\dot{x}^\nu(\cdot)$ converges to $\dot{x}(\cdot)$ pointwise a.e. on $[t_0,t_1]$. □

Continuity questions are not the only ones to consider. Much of the actual work done in control applications is concerned with *linear* control systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ for a.e. } t \in [t_0,t_1], \quad x(t_0) = a,$$

both for practical and theoretical convenience, and the question of approximation of nonlinear systems by such systems is naturally raised. It is clear that one can always take a nonlinear expression $f(t,x,u)$ and replace it by a linear one like $A(t)x + B(t)u$, but what does this mean in terms of the mapping $\mathcal{F}$ associated with the control system? Is there a sense in which we can rigorously introduce such approximations as a form of "linearizing" $\mathcal{F}$? If so, what are the mathematical properties of such linearizations, and how reliable can we expect them to be?

In calculus, the idea of linearizing a function is closely connected with differentiation. A function on R can be approximated by replacing its graph around a certain point by the tangent line to the graph at that point, and in this way one obtains a linear function. It is not surprising that something quite analogous happens in our infinite-dimensional setting.

For the moment, let us just think of $\mathcal{F}$ as a mapping from a space W to another space X, both of these being linear spaces (vector spaces) supplied with norms. The norm in each case will simply be denoted by $||\cdot||$. Later we will take $W = R^n \times \mathcal{L}_d^\infty[t_0, t_1]$ and $X = \mathcal{A}_n^\infty[t_0, t_1]$.

The mapping $\mathcal{F} : W \to X$ is said to be *differentiable* at the point $w_0 \in W$ (in the sense of Fréchet) if the difference quotient mappings $\mathcal{G}_s$ defined for $s \neq 0$ by

$$\mathcal{G}_s(z) = s^{-1}[\mathcal{F}(w_0 + sz) - \mathcal{F}(w_0)] \text{ for } \; z \in W$$

converge to *uniformly on all bounded sets of* W to a mapping $\mathcal{G} : W \to X$ that is *linear* and moreover *continuous* in the norm to norm sense. The uniform convergence on bounded sets of W means that for any choice of $r > 0$ and $\varepsilon > 0$ there is a $\delta > 0$ such that, for all z satisfying $||z|| \leq r$ and s satisfying $0 \leq |s| < \delta$, one has $||\mathcal{G}_s(z) - \mathcal{G}(z)|| < \varepsilon$. The linear mapping G is then called the *derivative* of $\mathcal{F}$ at w_0, and the notation $\mathcal{F}'(w_0; z) = \mathcal{G}(z)$ gives the *directional derivative* of $\mathcal{F}$ at w_0 relative to z.

This concept gives a precise expression to the local approximation of $\mathcal{F}$ by its "first order expansion," a linearized mapping: We have

$$\mathcal{F}(w_0 + sz) \approx \mathcal{F}(w_0) + s\mathcal{G}(z)$$

and can write this legitimately as

$$\mathcal{F}(w) = \mathcal{F}(w_0) + G(w - w_0) + o(w - w_0),$$

where $o(w - w_0)$ denotes an expression with the property that

$$||w - w_0||^{-1}||o(w - w_0)|| \to 0 \;\text{ as }\; ||w - w_0|| \to 0.$$

The issue now is whether, under some reasonable assumptions on our control system function f, the control mapping $\mathcal{F} : R^n \times \mathcal{L}_d^\infty[t_0, t_1] \to \mathcal{A}_n^\infty[t_0, t_1]$ is differentiable. We shall show that it is, and that the derivative is the control mapping associated with a system obtained by linearizing f.

Theorem 2.6 *Consider a control system*

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ for a.e. } t \in [t_0, t_1], \quad x(t_0) = a,$$

in which the function f is differentiable in u as well as x, and the Jacobians $\nabla_x f(t, x, u)$ and $\nabla_u f(t, x, u)$ as well as $f(t, x, u)$ itself all depend continuously on (t, x, u). Assume also the growth property that:

> *For any bounded set $U_0 \subset R^d$ there is a constant μ such that $|f(t, x, u)| \leq \mu(1 + |x|)$ for a.e. $t \in [t_0, t_1]$ and every $x \in R^n$ when $u \in U_0$.*

Then the mapping $\mathcal{F} : R^n \times \mathcal{L}_d^\infty[t_0, t_1] \to \mathcal{A}_n^\infty[t_0, t_1]$ with $\mathcal{F}(a, u(\cdot)) = x(\cdot)$ is not only well defined by the standards of Theorem 2.4, but also differentiable at every pair $(a, u(\cdot))$. Its derivative at $(a, u(\cdot))$ is the mapping $\mathcal{G} : R^n \times \mathcal{L}_d^\infty[t_0, t_1] \to \mathcal{A}_n^\infty[t_0, t_1]$ with $\mathcal{G}(b, v(\cdot)) = y(\cdot)$, where $y(\cdot)$ is the trajectory corresponding to the initial state b and the control function $v(\cdot)$ in the linear system

$$\dot{y}(t) = A(t)y(t) + B(t)v(t) \text{ for a.e. } t \in [t_0, t_1], \quad y(t_0) = b$$

with

$$A(t) = \nabla_x f(t, x(t), u(t)) \quad \text{and} \quad B(t) = \nabla_u f(t, x(t), u(t)),$$

$x(\cdot)$ being the trajectory $\mathcal{F}(a, u)$.

Proof. The continuity and differentiability assumptions made here on f are stronger than the ones in Theorem 2.4. They ensure certainly that f satisfies the Carathéodory conditions and has $|\nabla_{x,u} f(t, x, u)|$ bounded above relative to any bounded set $[t_0, t_1] \times X_0 \times U_0$. Thus the hypothesis of Theorem 2.4 is fulfilled and the mapping $\mathcal{F}$ is well defined.

Fix any pair $(a, u(\cdot))$ in $R^n \times \mathcal{L}_d^\infty[t_0, t_1]$ and let $x(\cdot) = \mathcal{F}(a, u(\cdot))$. Consider arbitrary $b \in R^n$ and $v(\cdot) \in \mathcal{L}_d^\infty[t_0, t_1]$, fixed for the time being, and for each $s \neq 0$ define $x_s(\cdot)$ to be the trajectory $\mathcal{F}(a + sb, u(\cdot) + sv(\cdot))$, i.e. the unique solution to

$$\dot{x}_s(t) = f(t, x_s(t), u(t) + sv(t)) \text{ for a.e. } t \in [t_0, t_1], \quad x_s(t_0) = a + sb.$$

Further define $y_s(t) = s^{-1}[x_s(t) - x(t)]$, so that

$$y_s(\cdot) = s^{-1}[\mathcal{F}(a + sb, u(\cdot) + sv(\cdot)) - \mathcal{F}(a, u(\cdot))].$$

Our task is to show that $y_s(\cdot)$ converges in the space $\mathcal{A}_n^\infty[t_0, t_1]$ to the trajectory $y(\cdot)$ generated by the linearized system: $y(\cdot) = \mathcal{G}(b, v(\cdot))$. This convergence must be demonstrated moreover to be uniform in the sense that for arbitrary choice of $r > 0$ and $\varepsilon > 0$ we can find a $\delta > 0$ *not depending on the particular* b *and* $v(\cdot)$ *as long as* $|b| \le r$ *and* $||v(\cdot)||_\infty \le r$, such that

$$|||y_s(\cdot) - y(\cdot)|||_\infty < \varepsilon \text{ when } 0 < |s| < \delta.$$

Recall here that

$$|||y_s(\cdot) - y(\cdot)|||_\infty = \max\{|y_s(t_0) - y(t_0)|, ||\dot{y}_s(\cdot) - \dot{y}(\cdot)||_\infty\}.$$

¿From the definitions we have

$$y_s(t_0) - y(t_0) = s^{-1}[x_s(t_0) - x(t_0)] - y(t_0) = s^{-1}[a + sb - a] - b = 0$$

and

$$\begin{aligned}\dot{y}_s(t) - \dot{y}(t) \;&=\; s^{-1}[\dot{x}_s(t) - \dot{x}(t)] - \dot{y}(t)\\ &+s^{-1}[f(t, x_s(t), u(t) + sv(t)) - f(t, x(t), u(t))] - [A(t)y(t) + B(t)v(t)]\\ &+s^{-1}[f(t, x(t) + sy_s(t), u(t) + sv(t)) - f(t, x(t), u(t))]\\ &\qquad -\nabla_x f(t, x(t), u(t))y(t) - \nabla_u f(t, x(t), u(t))v(t).\end{aligned}$$

Let us introduce the function $z_s(t) = y_s(t) - y(t)$. We have $z_s(t_0) = 0$, so that

$$|||y_s(\cdot) - y(\cdot)|||_\infty = |||z_s(\cdot)|||_\infty = ||\dot{z}_s(\cdot)||_\infty.$$

It is the latter, then, whose convergence to 0 must be ascertained.

Define

$$\begin{aligned}g_s(t, z) \;&=\; s^{-1}[f(t, x(t) + sy(t) + sz, u(t) + sv(t)) - f(t, x(t), u(t))]\\ &\qquad -\nabla_x f(t, x(t), u(t))y(t) - \nabla_u f(t, x(t), u(t))v(t).\end{aligned}$$

In terms of this we have the differential equation

$$\dot{z}_s(t) = g_s(t, z_s(t)) \text{ a.e. } t \in [t_0, t_1], \quad \text{with } z_s(t_0) = 0.$$

An analysis of the properties of this equation will produce for us the desired conclusion that $||\dot{z}_s(\cdot)||_\infty \to 0$ and does so with a certain uniformity relative to the choice of the underlying elements b and $v(\cdot)$.

The main fact to invoke now is the continuity of the first derivatives of f in x and u with respect to all the variables, including t. This implies by the classical calculus of functions of several variables (via the mean value theorem) that the difference functions

$$\begin{aligned} h_s(t,x,u,y,v) &= s^{-1}[f(t,x+sy,u+sv)-f(t,x,u)] \\ &\qquad -\nabla_x f(t,x,u)y - \nabla_u f(t,x,u)v \end{aligned}$$

converge to 0 as $s \to 0$ and do so *uniformly over any bounded set of points* (t,x,u,y,v). It follows in our setting of functions $x(\cdot)$, $u(\cdot)$, $y(\cdot)$, and $v(\cdot)$ that for any choice of $r > 0$ and $\varepsilon > 0$ there will be a $\delta > 0$ such that

$$\begin{aligned} &|h_s(t,x(t),u(t),y(t)+z,v(t))| < \varepsilon \text{ for a.e. } t \in [t_0,t_1] \\ &\qquad \text{when } 0 < |s| < \delta,\ |z| \le r, |b| \le r, \text{ and } ||v(\cdot)||_\infty \le r. \end{aligned}$$

The norm on b enters in because $y(\cdot)$ depends on b as well as $v(\cdot)$; but the vectors $y(t)$ will remain in a bounded region of R^n independent of the particular choice of b and $v(\cdot)$ as long as $|b| \le r$ and $||v(\cdot)||_\infty \le r$ for some fixed r. Again one can make use of Proposition 2.3. For this linear differential equation, one has

$$|\dot{y}(t)| \le |A(t)||y(t)| + |B(t)||v(t)| \text{ for a.e. } t \in [t_0,t_1].$$

(In Proposition 2.3 one can take $\mu_0(t) = r|B(t)|$, $\mu_1(t) = |A(t)|$, and $\alpha = r$ to establish the desired bound.)

Of course $g_s(t,z) = h_s(t,x(t),u(t),y(t)+z,v(t))$, so we get

$$\begin{aligned} &|g_s(t,z)| < \varepsilon \text{ for a.e. } t \in [t_0,t_1] \\ &\qquad \text{when } 0 < |s| < \delta,\ |z| \le r, |b| \le r, \text{ and } ||v(\cdot)||_\infty \le r. \end{aligned}$$

Under this property we find that the function $z_s(\cdot)$ must, by virtue of the differential equation it satisfies, have $|\dot{z}_s(t)| < \varepsilon$ over any interval $[t_0, t_0+\tau]$ on which $|z_s(t)| \le r$. Over such an interval it then has

$$|z_s(t)| = |\int_{t_0}^{t} \dot{z}_s(\tau)d\tau| < \varepsilon(t-t_0),$$

because $z_s(t_0) = 0$. Obviously, therefore, after fixing any $r > 0$ and taking ε small enough that $\varepsilon(t_1 - t_0) < r$, we will be able to select $\delta > 0$ such that, regardless of the particular b and $v(\cdot)$ as long as $|b| \le r$ and $||v(\cdot)||_\infty \le r$, we will have

$$|z_s(t)| \le r \text{ for all } t \in [t_0,t_1] \text{ when } 0 < |s| < \delta.$$

Then

$$|\dot{z}_s(t)| \le \varepsilon \text{ for a.e. } t \in [t_0, t_1] \text{ when } 0 < |s| < \delta.$$

In other words, after fixing any $r > 0$, we can find for given any $\varepsilon > 0$ a $\delta > 0$ such that, regardless of the particular b and $v(\cdot)$ as long as $|b| \le r$ and $||v(\cdot)||_\infty \le r$, we will have

$$||\dot{z}_s(\cdot)||_\infty \le \varepsilon \text{ when } 0 < |s| < \delta.$$

This is just what we needed to prove. □

Corollary 2.7 *Under the assumptions in Theorem 2.6 the mapping* $\mathcal{F} : R^n \times \mathcal{L}_d^\infty[t_0, t_1] \to \mathcal{A}_n^\infty[t_0, t_1]$ *is in particular continuous everywhere relative to the norms on these spaces.*

Proof. The definition of differentiability (in the sense of Fréchet) implies continuity in the norms, as can readily be verified. □

Note that even though the function f is assumed in Theorem 2.6 to be continuous with respect to t, the linearized equation that one gets will *not* necessarily have $A(t)$ and $B(t)$ continuous in t. This is of course due to the fact that the control function $u(\cdot)$ at which the linearization is taken might not be continuous in t.

The concept of linearization made rigorous in Theorem 2.6 is especially important in treating problems of keeping a system in or near equilibrium. In the context already discussed in a preliminary way in Section 2 of Chapter 1, let us imagine an autonomous nonlinear control equation with the property that for a certain state $\bar{x}_0$, an *equilibrium* state, one has

$$f(\bar{x}_0, 0) = 0.$$

Then the perturbed trajectory $x(\cdot)$ emanating from an initial state $\bar{x}_0 + b$ near to $\bar{x}_0$ can be approximated by $x(t) \approx \bar{x}_0 + y(t)$, where $y(\cdot)$ is obtained by solving

$$\dot{y}(t) = Ay(t) + Bu(t) \text{ for a.e. } t \in [t_0, t_1] \;\text{ and }\; y(t_0) = a$$

where $A = \nabla_x f(\bar{x}_0, 0)$ and $B = \nabla_u f(\bar{x}_0, 0)$. Characteristics of the matrices A and B will largely determine the modes of control as long as the perturbations remain small.

Needless to say, when perturbations become large this kind of local linearization could fail to be effective in determining how to restore the system to equilibrium. The key ingredient to any estimate of the range of effectiveness of the linearization would be an analysis of the actual rates of uniform convergence referred to in the definition of the differentiability of the mapping $\mathcal{F}$. The specific foundations for this are laid out in the proof of Theorem 2.6.