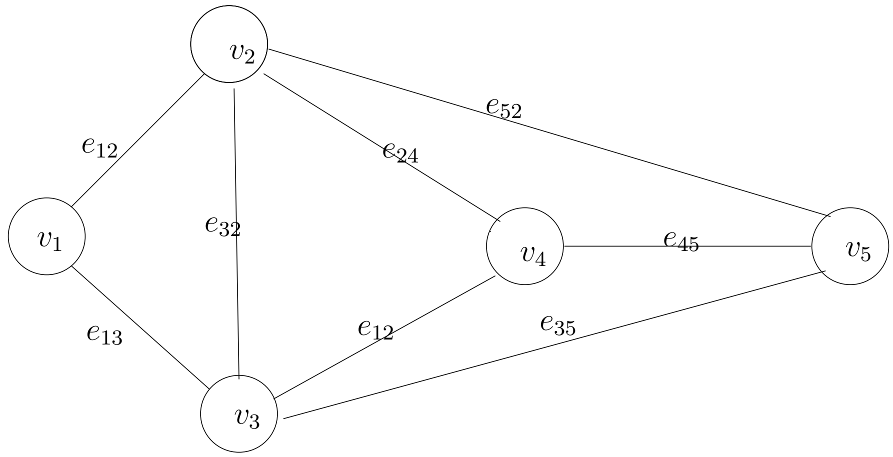

FIGURE 1. A graph with 5 vertices.

1. Graphs, Digraphs, and Networks

1.1. **The Basics.** A graph is a mathematical structure comprised of two classes of objects: vertices and edges. If we let G denote the graph, then we write $G = (V, E)$ where V is the set of vertices and E the set of edges. The edges E are a subset of $V \times V$ consisting of unordered pairs of vertices. If $v_1, v_2 \in V$ and $e = \{v_1, v_2\} \in E$, then we say v_1 and v_2 are adjacent to each other in G, and that e joins v_1 to v_2 and e is incident to both v_1 and v_2. It is useful to think of networks pictorially as in the figure below where the circles represent the vertices (v_1, v_2, v_3, v_4, v_5) and the lines connecting the circles are the edges.

The edges in this graph are

$$\{v_1, v_2\},\ \{v_2, v_3\},\ \{v_1, v_3\},\ \{v_2, v_5\},\ \{v_2, v_4\},\ \{v_4, v_3\},\ \{v_5, v_3\},\ \{v_4, v_5\}.$$

A graph is said to be a *digraph* (or *network*) if the pairs of vertices that determine the edges are ordered. In the case we write $e = (v_i, v_j)$ to emphasize that v_i comes first and v_j second. In a digraph the vertices are called nodes and edges arcs. A digraph version of the graph given above follows.

The edges in this graph are

$$(v_1, v_2),\ (v_1, v_3),\ (v_3, v_2),\ (v_2, v_4),\ (v_5, v_2),\ (v_4, v_5),\ (v_4, v_3),\ (v_3, v_5).$$

In a digraph these pairs of vertices are still said to be adjacent, but now we can be more specific about their adjacency. If (v_i, v_j) is an edge in a digraph, we say that v_i is adjacent *to* v_j and v_j is adjacent *from* v_i. Similarly, we say that the arc (v_i, v_j) is incident from v_i and incident to v_j.

Any pair of edges between the same pair of vertices are said to be parallel edges, and any edge from a vertex to itself is called a loop. A graph is said to be *simple* if it has no loops or parallel edges.

The graph $G = (V, E)$ is said to be *bipartite* if the vertex set can be partitioned into two sets X and Y such that $\{v_i, v_j\} \in E$ if and only if either $v_i \in X$ and $v_j \in Y$, or $v_j \in X$ and $v_i \in Y$. In this case we write $G = (X, Y, E)$.

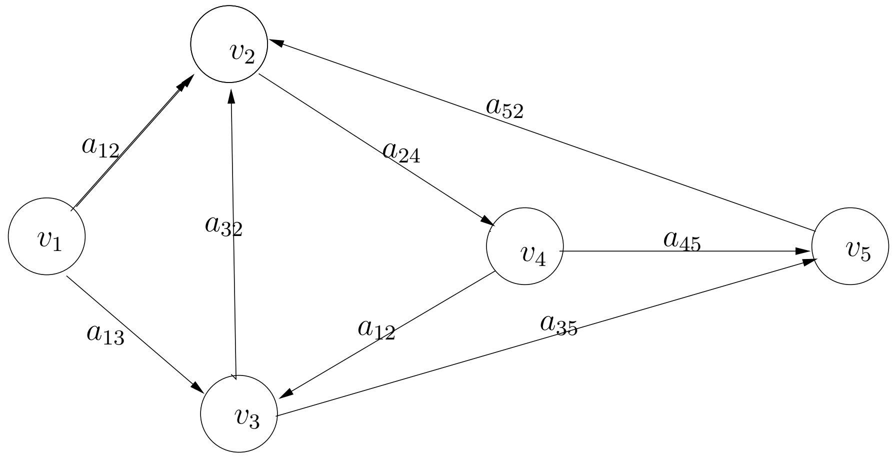

FIGURE 2. A digraph with 5 nodes.

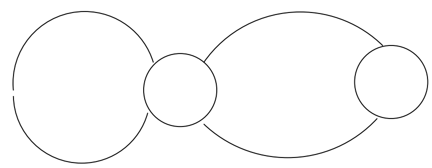

FIGURE 3. Loops and parallel edges.

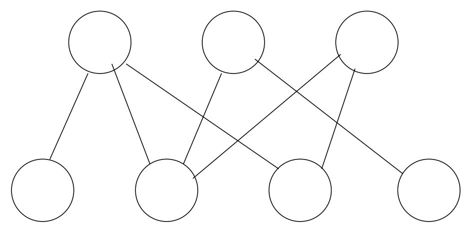

FIGURE 4. A bipartite graph.

The *complete graph* on n vertices, denoted K_n is the simple graph having all vertices adjacent to each other. The *complete bipartite graph* $K_{r,s} = (X, Y, E)$ is the bipartite graph where every element of X is adjacent to every element of Y with $|X| = r$ and $|Y| = s$.

A graph $G = (V, E)$ is said to be *cyclic* if its distinct vertices have and ordering $\{v_1, v_2, \ldots, v_n\}$ such that $E = \{\{v_1, v_2\}, \ \{v_2, v_3\}, \ \{v_3, v_4\}, \ldots, \{v_{(n-1)}, v_n\}, \ \{v_n, v_1\}\}$, where $n = |V| \geq 3$. We denote the cyclic graph on n vertices by C_n.

The graph $H = (W, F)$ is said to be a subgraph of the graph $G = (V, E)$ if $W \subset V$ and $F \subset E$. The subgraph is said to be spanning if $W = V$. The graph $R = (U, T)$ is said to be a super-graph of $G = (V, E)$ if $V \subset U$ and $E \subset T$.

Consider the graph $G = (V, E)$ and label the vertices by $V = \{v_1, v_2, \ldots, v_n\}$. The vertex-edge incidence matrix for G corresponding to this labeling is an $n \times m$ matrix A where $n = |V|$ and $m = |E|$. The ith row of A corresponds to the vertex v_i, and each column

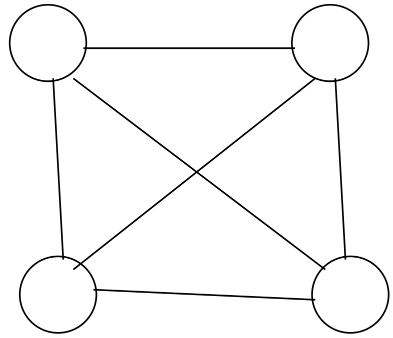

FIGURE 5. The compete graph K_4.

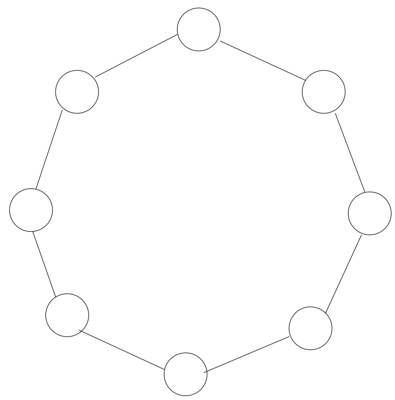

FIGURE 6. The cyclic graph C_8.

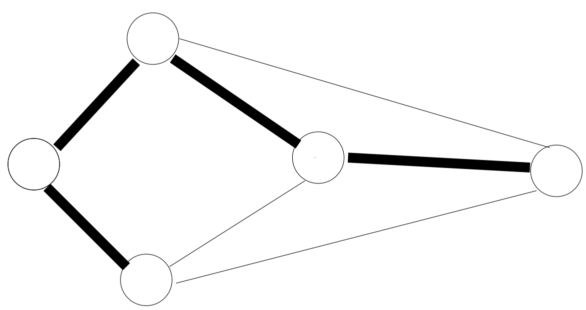

FIGURE 7. The bold edges give a spanning subgraph.

corresponds to an edge (also ordered in some way).

$$\text{If } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ is the column corresponding to the edge } e = \{v_i, v_j\}, \text{ then}$$

$$a_i = 1 = a_j \text{ and } a_k = 0 \text{ for all } i \neq k \neq j \ .$$

For example, the vertex-edge incidence matrix for the graph in figure (1) is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

If $G = (V, E)$ is a digraph with labeled nodes $V = \{v_1, v_2, \ldots, v_n\}$, then the node-arc incidence matrix differs from the $vertex - edge$ incidence matrix by distinguishing whether the an arc is adjacent from or to a vertex.

$$\text{If } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ is the column corresponding to the edge } e = (v_i, v_j), \text{ then}$$

$$a_i = -1,\ a_j = 1 \text{ and } a_k = 0 \text{ for all } i \neq k \neq j\ .$$

For example, the vertex-edge incidence matrix for the graph in figure (2) is

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}.$$

Note that every column of either a vertex-edge or a node-arc incidence matrix contains exactly two nonzero entries, and the sum of the entries in every column of a node-arc incidence matrix is zero.

The degree of a vertex in a graph is the number of distinct edges incident to it. The out-degree of a node in a digraph is the number of distinct edges incident from the vertex and the in-degree is the number of edges incident to the vertex. Note that the row sum of the entries in a vertex-edge incidence matrix equals the degree of the vertex associated with that row, where as the row sum of the entries in a node-arc incidence matrix equals the in-degree minus the out-degree of the vertex associated with that row.

Exercise: What are the degrees of all vertices in the graph (1)?

Exercise: What are the in-degrees and the out-degrees of all notes in the digraph (2)?

Exercise: Show that the sum of all of the degrees of all of the vertices in a graph equals two times the number of edges.

Exercise: Show that the number of vertices in graph having odd degree is even.

Exercise: Show that the sum of all of the out-degrees of all of the vertices in a graph equals the sum of in-degrees of all of the vertices in the graph which equals the number of edges in the graph.

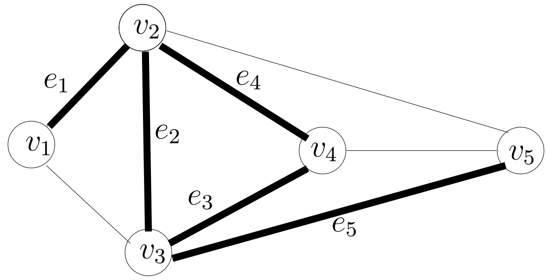

FIGURE 8. A walk from v_1 to v_5.

1.2. **Connectivity.** Let $G = (V, E)$ be a graph and $v, w \in V$. A *walk* between v and w is a finite alternating sequence

$$v = v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k = w \ ,$$

where $e_s = \{v_{s-1}, v_s\},\ s = 1, \ldots, k$. All of the edges and vertices in the sequence other than v and w are called intermediary. The vertices and edges in a walk need not be distinct. Two walks are said to be equivalent if they are given by exactly the same alternating sequence of vertices and edges. The number of edges in a walk is called the length of the walk. A walk in a digraph is said to be directed if $e_s = (v_{s-1}, v_s),\ s = 1, \ldots, k$.

A walk is a called a *path* if all of the edges are distinct. A walk in a digraph is said to be *directed path* if it is a directed walk and a path.

Theorem 1.1. *Let v and w be two distinct vertices in a graph G. The every walk between v and w contains a path from v to w where no two vertices are repeated.*

Proof. Let

$$v = v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k = w \tag{1}$$

be the walk between v and w. If there does not exist a vertex in $\{v_0, v_2, \ldots, v_k\}$ that is repeated, then the walk (1) must be a path with no repeating vertices since in this case all of the vertices in the path are distinct. Hence we can assume that there is a smallest index i_1 such that $v_{i_1} \in \{v_0, v_1, \ldots, v_{i_1-1}\}$. Suppose $v_j = v_{i_1}$ with $j < i_1$ so that the vertices $v_0, v_1, \ldots, v_j$ are distinct. Cut the vertices and edges $e_{j+1}, v_{j+1}, \ldots, v_{i_1}$ from the walk (1) to create the walk

$$v = v_0, e_1, v_1, e_2, v_2, \ldots, v_j, e_{i_1+1}, v_{i_1+1}, \ldots, v_k = w$$

from v to w where all of the vertices $v_0, v_1, \ldots, v_j$ are distinct since all of these indices are smaller that i_1. Repeat this process by letting $i_2(> i_1 > j)$ be the smallest index such that $v_{i_2} \in \{v_0, v_1, \ldots, v_{i_2-1}\}$ (again, if no such index exists, we are done). Since the walk is of finite length and $v_{i_s} < v_{i_{s+1}}$ for all s, this process can only be repeated finitely many times before the trimmed walk contains no repeated vertices. When this occurs, we have a walk with no repeated vertices, and hence no repeated edges. That is, the walk is now a path with no repeated vertices. $\square$

Corollary 1.1.1. *Let v and w be two distinct vertices in a digraph G. Then every directed walk between v and w contains a directed path from v to w where no two vertices are repeated.*

Exercise: Prove Corollary 1.1.1.

The walk (1) is said to be closed if $v = w$. A circuit is a closed walk with no repeated edges, and a cycle is a circuit with no repeated intermediary vertices.

Proposition 1.1.1. *A subgraph C of a graph G is a cycle in G if and only if C is a cyclic graph.*

Exercise: Give an example of a closed walk that does not contain a circuit.

Theorem 1.2. *Every circuit in a graph contains a cycle.*

Proof. Suppose (1) is a circuit and let $\hat{k}$ be the smallest index such that $v_{\hat{k}} \neq v_{\hat{k}+1} = v_{\hat{k}+2} = \cdots = v_k$ (note that the edges $e_{\hat{k}+2}, \ldots, e_k$ are distinct loops at v_k). Then $v_0 \neq v_{\hat{k}}$ and

$$v = v_0, e_1, v_1, e_2, v_2, \ldots, v_{\hat{k}}$$

is a walk from v_0 to $v_{\hat{k}}$. Then, by Theorem 1.1, this walk contains a path P from v to $v_{\hat{k}}$ in which no vertex is repeated. Since (1) is a circuit, the edge $e_{\hat{k}+1}$ between $v_{\hat{k}}$ and v_0 cannot appear in the path from v_0 to $v_{\hat{k}}$, hence the circuit obtained by adjoining the edge $e_{\hat{k}+1}$ to the path P is a cycle. $\square$

A cycle is said to be even or odd if its length is even or odd, respectively. A directed circuit is a directed path that is also a circuit, and a directed cycle is a cycle that is also a directed path.

Corollary 1.2.1. *Every directed circuit contains a directed cycle.*

Exercise: Prove Corollary 1.2.1.

Two vertices in a graph are said to be connected if there is a path between them. A graph is said to be connected if every pair of vertices in it are connected. Two vertices in a digraph are said to be strongly connected if there is a directed path between them. A digraph is said to be strongly connected if every pair of vertices in it are strongly connected. If a digraph is connected, but not strongly connected, we say it is weakly connected.

A subgraph H of the graph G is said to be component of G if H is connected and the only connected subgraphs of G that contain H are G and H. Obviously, a connected graph has only one component.

Let $G = (V, E)$ be a graph. A set $F \subset E$ is said to a disconnecting set of edges if the graph $(V, E \setminus F)$ has more components than G. If $F = \{f\}$ is a disconnecting set, we call the edge f a bridge. A disconnecting set F is called a cut set if it contains no proper subset of edges that is disconnecting.

Lemma 1.1. *If C is a cycle in a connected graph $G = (V, E)$ and e is any edge in the cycle, then the the graph $G' = (V, E \setminus \{e\})$ is still connected.*

Exercise: Prove Lemma 1.1.

1.3. **Forests and Trees.** An acyclic graph is a graph with no cycles. Such a graph is a called a forest. A tree is a connected acyclic graph. Hence, the connected components of a forest are trees. An acyclic spanning subgraph is called a spanning forest, and a acyclic connected spanning subgraph is called a spanning tree. Clearly, if a graph has a spanning tree, then it must be connected. Soon we will see that the converse is also true, that is, every connected graph has a spanning tree.

The notions of a tree and a spanning tree will be important to our study. Hence we begin with a careful study of their properties.

Theorem 1.3. *The following are equivalent in a graph $G = (V, E)$ with n vertices.*

(1) *G is a tree.*
(2) *There is a unique path between every pair of vertices in G.*
(3) *G is connected, and every edge in G is a bridge.*
(4) *G is connected, and it has $(n-1)$ edges.*
(5) *G is acyclic, and it has $(n-1)$ edges.*
(6) *G is acyclic, and whenever any two vertices in G are joined by an edge, the resulting enlarged graph has a unique cycle.*
(7) *G is connected, and whenever any two vertices in G are joined by an edge, the resulting enlarged graph has a unique cycle.*

Proof. (1)$\Rightarrow$(2): Suppose to the contrary that there exist two distinct vertices v and w having two distinct paths P and Q between them. Let $e = \{v_i, v_{i+1}\}$ be the first distinct edge between P and Q as the paths progress toward w, and let the vertex v_j be the first common vertex between P and Q after v_i. Then the path along P from v_i to v_j, then back along Q from v_j to v_i is a cycle in the acyclic graph G. Hence, there is a unique path between every pair of distinct vertices.

(2)$\Rightarrow$(3): Since there is a unique path between each pair of vertices, G is connected. Given any edge $e = \{v_i, v_{i+1}\}$, the sequence v_i, e, v_{i+1} is the unique path from v_i to v_{i+1}, so deleting this edge destroys the only path from v_i to v_{i+1}. Hence e is a bridge.

(3)$\Rightarrow$(4): We need only show that G has $(n-1)$ edges. We proceed by induction on the number of vertices. The result is clearly true for $|V| = 1$. Assume it is true for $1 \leq |V| \leq k-1$, and show it is true for $|V| = k$. Let e be an edge of G. Since it is a bridge, removing it gives two connected graphs G_1 and G_2 where each edge is a bridge. Let k_1 and k_2 be the number of vertices in G_1 and G_2, respectively. In particular, $k_1 < k$, $k_2 < k$, and $k = k_1 + k_2$. The induction hypothesis tells us that G_1 has $k_1 - 1$ edges and G_2 has $k_2 - 1$ edges. Hence the total number of edges in G must be $(k_1 - 1) + (k_2 - 1) + 1 = k_1 + k_2 - 1 = k - 1$, which proves the implication.

(4)$\Rightarrow$(1): We need only show that G is acyclic. Assume to the contrary that G contains a cycle C. Let G' be the subgraph of G obtained by removing any edge in C. Since C is a cycle, G is still connected. Continue removing edges in this way, until no cycles remain. Denote the resulting graph by G''. Then G'' is a connected acyclic graph, that is, G'' is a tree. This tree has n vertices (since we removed none) and $k < (n-1)$ edges since we started with $(n-1)$ and we removed at least one edge from the first cycle. But by what we have show thus far, (1)$\Rightarrow$(4), so G'' must have $(n-1)$ edges! This contradiction implies that the cycle C cannot exist. That is, G is a tree as required.

(4)$\Rightarrow$(5): Since (4) implies that G is a tree, G is acyclic.

(1)$\Rightarrow$((6) or (7)): We need only show that whenever any two vertices in G are joined by an edge, the resulting enlarged graph has a unique cycle. First observe that since (1)$\Rightarrow$(4), G has $(n-1)$ vertices. Now, suppose to the contrary that there is a pair of vertices v and w such that adding the edge $e = \{v, w\}$ creates two distinct cycles C_1 and C_2 in the resulting enlarged graph $G' = (V, E \cup \{e\})$. Since C_1 and C_2 are distinct cycles, C_1 contains an edge f_1 that is not in C_2. Therefore, deleting f_1 does not effect the cycle C_2 and, since f_1 is in a cycle, the resulting graph $G'' = (V, (E \cup \{e\}) \setminus \{f_1\})$ is connected and contains the cycle C_2. Therefore, we can pick any edge f_2 in C_2 and delete it from G'' and have the resulting graph $G''' = (V, (E \cup \{e\}) \setminus \{f_1, f_2\})$ remain connected. But G'' is a connected graph with $(n-1)$ edges (since G has $(n-1)$). Hence, since (4)$\Rightarrow$(1), G'' is a tree so that every edge of G'' is a bridge. But we removed f_2 from G'' to get the connected graph G'''. This contradiction establishes the result.

(6)$\Rightarrow$(1): We need only show that the graph is connected. If it were not connected, then it must contain at least two components G_1 and G_2, both of which are necessarily acyclic. Let v_1 be a vertex in G_1 and v_2 a vertex in G_2. If we add now the edge $e = \{v_1, v_2\}$, we must obtain a cycle, and e must be part of this cycle since no cycle previously existed. Being part of cycle implies that there is a path from v_1 to v_2 that does not contain e. But since v_1 and v_2 lie in separate components of G, no such path can exist. Hence G must be connected.

(7)$\Rightarrow$(1): We need only show that the graph is acyclic. If G contains only one vertex, the result is obvious, so assume G contains at least two vertices. Since G is connected, there exists a pair of adjacent vertices v_1 and v_2 with connecting edge e. By adding a second edge f between these vertices, we obtain the unique cycle (v_1, e, v_2, f, v_1). Hence if we delete f the resulting graph, namely G, must be acyclic. $\square$

Finally, we can show that every connected graph possess a subgraph that is a spanning tree.

Theorem 1.4. *A graph is connected if and only if it contains a spanning tree.*

Proof. Obviously, if a graph has a spanning tree, then it must be connected. Hence we assume that G is connected and show that it has a spanning tree. The strategy is to remove all cycles in G. If G is already acyclic, then G itself is a tree. So we can assume that G contains a cycle C_1. If we remove any edge e_1 from C_1, then the resulting graph $G_1 = (V, E \setminus \{e_1\})$ is still connected. If G_1 is acyclic, then we are done; otherwise, repeat this process. Obviously, this process can only be repeated finitely many time since there are only finitely many edges. Hence, the process must eventually terminate with a connected graph $G_k = (V, E \setminus \{e_1, e_2, \ldots, e_k\})$ that is acyclic. Since it is spanning, it is a spanning tree. $\square$

1.4. **Incidence Matrices in a Digraph.** We now connect trees in a digraph to node-arc incidence matrices. Recall that in a simple graph there are no loops or parallel arcs.

Theorem 1.5. *Let $G = (V, E)$ be a (weakly) connected digraph with $n = |V|$, $m = |E|$, and $n \times m$ node-arc incidence matrix A. Then the rank of A is $(n-1)$. That is, every $k < n$ rows of A are linearly independent.*

Proof. Let $A = (a_{ij})$, where A_{ij} denotes the ij entry of A. Since every column of A contains exactly one 1 and one -1 with all other entries being zero, we have $\mathsf{e}^T A = 0$, where e is the n-vector of all ones. Hence the rank of A is less than n. If the result were false, there would exist $1 \le k \le n-1$ rows of A, $i_1, i_2, \dots, i_k$, that were linearly dependent. Clearly, $k > 1$ since the connectivity of G implies that every row of A contains at least one non-zero entry. Let $A_{i_s\cdot}$ denote the i_sth row of A, $s = 1, \dots, k$, and let $z \in \mathbb{R}^k$ be such that $0 = \sum_{s=1}^k z_s A_{i_s\cdot}$, or equivalently, $0 = \sum_{s=1}^k z_s a_{i_s j},\ j = 1, \dots, m$. With no loss in generality, we may assume that $z_s \neq 0,\ s = 1, \dots, k$. Since every z_s is non-zero, each of the sets $\{a_{i_1 j}, a_{i_2 j}, \dots, a_{i_k j}\}$ has either two non-zero elements, or all of the elements are zero. But then the vertices $i_1, i_2, \dots, i_k$ and the edges associate with the nonzero sets $\{a_{i_1 j}, a_{i_2 j}, \dots, a_{i_k j}\}$ necessarily form a component of G. Since G is connected, this implies the contradiction $n = k < n$. □

Corollary 1.5.1. *Let $G = (V, E)$ be a digraph with $n = |V|$ and $n - 1 = |E|$. Then G is weakly connected if and only if the rank of its node-arc incidence matrix is $n - 1$.*

Proof. The theorem shows that if G is weakly connected, then the rank of the node-arc incidence matrix A is $n-1$. Let us now suppose that the rank is $n-1$ with $|E| = n-1$ and show that G is connected. Assume on the contrary that it is not connected, and let $G_s,\ s = 1, \dots, k$ be the connected components of A. Permute the vertices and edges of G if necessary so that A has the form

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix},$$

where A_s is the node-arc incidence matrix for $G_s,\ s = 1, \dots, k$. Since each G_s is a connected digraph, the theorem tells us that each A_s has rank $n_s - 1$ where n_s is the number of vertices in G_s, $s = 1, \dots, k$. Thus, since A is block diagonal, the rank of A is the sum of the ranks of the A_s's, i.e., $(n-1) = \text{rank}(A) = \sum_{s=1}^k (n_s - 1) = n - k < (n-1)$. This contradiction implies that A is connected. □

Given a node-arc incidence matrix, a reduced incidence matrix is obtained by eliminating any of its rows.

Theorem 1.6. *Let $G = (V, E)$ be a (weakly) connected digraph with $n = |V|$, $m = |E|$, and $n \times m$ node-arc incidence matrix A. Let A_r be any reduced incidence matrix for G. Then an $(n-1) \times (n-1)$ submatrix B of A_r is nonsingular if and only if the edges corresponding to the columns of B are the edges of a spanning tree for G.*

Proof. Let B be an $(n-1) \times (n-1)$ submatrix B of A_r, and let $G' = (V, F)$ be the subgraph of G whose edges F correspond to the columns of B. Then, by Corollary 1.5.1, B is nonsingular if and only if G' is a connected graph with $(n-1)$ edges. By Theorem 1.3 Part (4), this is equivalent to G' being a tree, or equivalently, G' is a spanning tree for G. □

Theorem 1.7. *Let $G = (V, E)$ be a (weakly) connected digraph with $n = |V|$, $m = |E|$, and $n \times m$ node-arc incidence matrix A. Then A is totally unimodular.*

Exercise: Prove Theorem 1.7.

Theorem 1.8. *Let $G = (V, E)$ be a (weakly) connected digraph with $n = |V|$, $m = |E|$, and $n \times m$ node-arc incidence matrix A. Let A_r be any reduced incidence matrix for G. Then the number of spanning trees in G equals $\det(A_r A_r^T)$.*

1.5. **Bipartite Graphs.** Recall that a graph $G = (V, E)$ is said to be bipartite if the vertex set V can be partitioned into two sets X and Y such that given any two adjacent vertices v and w in V either $v \in X$ and $w \in Y$, or $v \in Y$ and $w \in X$. In this case we write $G = (X, Y, V)$.

Theorem 1.9. *A graph is bipartite if and onl if each of its components is bipartite.*

Exercise: Prove Theorem 1.9.

Lemma 1.2. *If a graph is bipartite, then every cycle in the graph is even.*

It is a remarkable fact that for a simple graph every cycle being even implies that a graph is bipartite.

Theorem 1.10. *Let G be a simple graph each of whose components has 3 or more vertices. Then G is bipartite if an only if it has no odd cycles.*

Proof. By Lemma 1.2, we need only show that if a graph each of whose components has 3 or more vertices has no odd cycles, then it is bipartite. To this end, let $G = (V, E)$ be graph each of whose components has 3 or more vertices and has no odd cycles. If we can show that each component is bipartite, then, by Theorem 1.9 we will have established the result. Hence, we may assume with no loss of generality that G is connected. On $V \times V$ define the function

$$d(v, w) = \text{minimum length of all paths connecting } v \text{ and } w.$$

We say that a path P from v to w is a shortest path, if its length equals $d(v, w)$. Let $u \in V$ and define

$$\begin{aligned} X &= \{x \in V \mid d(u, x) \text{ is even}\} \qquad \text{and} \\ Y &= V \setminus X\ . \end{aligned}$$

We will show that no two vertices in X (or Y) can be adjacent which implies that G is bipartite with $G = (X, Y, E)$. We do this in three steps.

Step (i): Show that u is not adjacent to any vertex in $X \setminus \{u\}$.

If $X \setminus \{u\} = \emptyset$, we are done; otherwise, let $v \in X \setminus \{u\}$. If $e \in E$ is an edge that joins u and v, then, by definition, $d(u, v) = 1$ which is not odd. Hence, there can be no edge joining u and v.

Step (ii): Show that no two vertices in X are adjacent.

Suppose there exists two distinct vertices $v, w \in X$ and an edge e in E joining them. By Step (i), neither v nor w can be u. Let P and Q be the shortest paths from u to v and from

u to w, respectively. Let the lengths of P and Q be $2r$ and $2s$, respectively. Note that e cannot be in either P or Q. Indeed, if $e \in P$, then a shortest path to v is to first use Q to get to w, since Q is a shortest path to w, and then go from w to v along e. But then $d(u,v) = d(u,w) + 1$ so that one of $d(u,w)$ and $d(u,v)$ is odd, a contradiction. A similar contradiction occurs if $e \in Q$, so e cannot be in either P or Q. Let u' be a vertex in both P and Q that is as far from u as possible so that the remaining portions of the paths P and Q from u' to v and u' to w, respectively, have no vertices in common. If $u' = v$, then $e \in Q$, so $u' \neq v$. Similarly, $u' \neq w$. If $u' = u$, adding e to P and the reverse of Q, both of which are even paths, gives and odd cycle, a contradiction. Hence $u' \neq u$. Now since the path P is shortest, it must be a shortest path from u at every vertex along the way. Hence the length of the path k from u to u' in P must be the same as the length of the path from u to u' in Q. Therefore, the length of the subpath P' from u' to V in P is $(2r - k)$ and the length of the subpath Q' from u' to w in Q is $(2s - k)$. Hence the cycle obtained by first using P' to get to v, then using e to get to w, then reversing Q' to go from w to u' has length $(2r - k) + (2s - k) + 1$ which is odd. This contradiction implies that no such edge e can exist. That is, no two vertices in X are adjacent.

Step (iii): Show that no two vertices in Y are adjacent.

Suppose to the contrary, that there are two distinct vertices $v, w \in Y$ that are joined be an edge $e \in E$. Proceed just as in Step (ii) and let P and Q be the shortest paths from u to v and from u to w, respectively, and set $2r + 1$ and $2s + 1$ equal to the lengths of P and Q, respectively. Again, e cannot be in either P or Q, else an odd cycle exists. Again let u' be a vertex in both P and Q that is as far from u as possible so that the remaining portions of the paths P and Q from u' to v and u' to w, respectively, have no vertices in common. As before, u' cannot be u, v, or w, else an odd cycle exists. Letting k be as defined in Step (ii), we find that the cycle obtained by first using P' to get to v, then using e to get to w, then reversing Q' to go from w to u' has length $(2r+1-k)+(2s+1-k)+1$ which is odd. This contradiction implies that no such edge e can exist. That is, no two vertices in Y are adjacent. $\square$

This result allows us characterize when an incidence matrix of a simple graph (not a digraph) is totally unimodular.

Theorem 1.11. *A simple graph is bipartite if and only if its incidence matrix is totally unimodular.*

Exercise: Show that the incidence matrix of a bipartite graph is totally unimodular.

Proof. We need only prove that if the simple graph G has an incidence matrix that is totally unimodular, then G must be bipartite. Let us suppose to the contrary that G is not bipartite. Then, by Theorem 1.10, G contains an odd cycle $v_1, e_1, v_2, e_2, \ldots, e_{2k-1}, v_{2k} = v_0$. Let the incidence matrix of G be arranged so that v_i corresponds to the ith row of A $i = 1, 2, \ldots 2k$ with all other vertices following, and e_j corresponds the j column of A, $j = 1, 2, \ldots, 2k - 1$, with all other edges following. Then the leading $(2k) \times (2k-1)$ submatrix of A has the form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & & & & \\ 0 & 1 & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & & & & 1 & 1 & 0 \\ 0 & & & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Let B be the $(2k-1) \times (2k-1)$ submatrix obtained by deleting the final row of this matrix. We compute the determinant of B by using Laplace's formula and expanding on the first row to get

$$\det(B) = \begin{vmatrix} 1 & 0 & & & & \\ 1 & 1 & & & & \\ 0 & 1 & 1 & & & \\ \vdots & & & \ddots & & \\ & & & 0 & 1 & 0 \\ & & \cdots & 0 & 1 & 1 \end{vmatrix} + (-1)^{2k} \begin{vmatrix} 1 & 1 & 0 & & & \\ 0 & 1 & 1 & & & \\ 0 & 0 & 1 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 1 & 1 \\ 0 & & & \cdots & 0 & 1 \end{vmatrix} = 1 + 1 = 2.$$

This contradiction implies that no such odd cycle can exist. Hence, G is bipartite. $\square$