

FIGURE 1. A graph with 5 vertices.

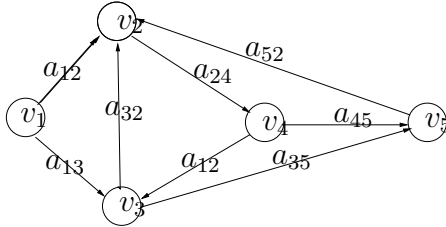


FIGURE 2. A digraph with 5 nodes.

## Problem Set 5: Graphs and Digraphs

- (1) What are the degrees of all vertices in the graph in Figure 1?
- (2) What are the in-degrees and the out-degrees of all nodes in the digraph in Figure 2?
- (3) Show that the sum of all of the degrees of all of the vertices in a graph equals two times the number of edges.
- (4) Show that the number of vertices of odd degree in a graph is even.
- (5) Let  $v$  and  $w$  be two distinct vertices in a digraph  $G$ . Show that every directed walk between  $v$  and  $w$  contains a directed path from  $v$  to  $w$  where no two vertices are repeated.
- (6) Give an example of a closed walk that does not contain a circuit.
- (7) Show that every directed circuit contains a directed cycle.
- (8) If  $C$  is a cycle in a connected graph  $G = (V, E)$  and  $e$  is any edge in the cycle, show that the graph  $G' = (V, E \setminus \{e\})$  is still connected.
- (9) Let  $G = (V, E)$  be a (weakly) connected digraph with  $n = |V|$ ,  $m = |E|$ , and  $n \times m$  node-arc incidence matrix  $A$ . Show that  $A$  is totally unimodular.
- (10) Show that a graph is bipartite if and only if each of its components is bipartite.
- (11) Show that the incidence matrix of a bipartite graph is totally unimodular.

The proofs of the total unimodularity of the incidence matrices of digraphs and bipartite graphs follows from Theorem 1.8 of the *Easy integer LP* notes. We restate this theorem below.

**Theorem 1.8:** Let  $A \in \mathbb{Z}^{m \times n}$  and let  $a_{ij}$  denote the  $ij$ th entry of  $A$ . If  $A$  satisfies the following three conditions, then  $A$  is totally unimodular.

- (1)  $a_{ij} \in \{0, \pm 1\}$  for all  $ij$ .

- (2) Every column of  $A$  has at most two non-zero entries.
- (3) The rows of  $A$  can be partitioned into two index sets  $I_1$  and  $I_2$  such that
  - (a) If a column has two entries of different signs, then the indices of the rows corresponding to these non-zero entries must be in the same index set.
  - (b) If a column has two entries of the same sign, then the indices of the rows corresponding to these non-zero entries must be in different index sets.

Vocabulary Words:

- (1) Graph, digraph, and bipartite graph.
- (2) Simple graphs.
- (3) Subgraphs.
- (4) Spanning subgraphs
- (5) Incidence matrices for graphs and digraphs
- (6) degree, in-degree, and out-degree of a vertex.
- (7) a walk, a path, a directed path
- (8) a closed walk, a circuit, and a cycle
- (9) directed paths and cycles
- (10) components of a graph
- (11) connected graphs
- (12) disconnecting sets, cuts, and bridges
- (13) an acyclic graph
- (14) forests, trees, and spanning trees
- (15) path length, odd and even paths and cycles