Problem Set 4: Totally Unimodular Matrices

(1) Which of the following matrices are totally unimodular. You must justify your answer.

(a)

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(f)

$$\begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

(2) Suppose $A \in \mathbb{R}^{m \times n}$ satisfies the conditions of Theorem 1.8. At the beginning of the proof of this theorem it is stated that *Clearly, due to the smallest degree requirement, the matrix B can have no column with a single non-zero entry.* Explain why this is true.

(3) Give an example of a totally unimodular matrix that does not satisfy the (1), (2), and (3) of Theorem 1.8.

(4) Prove formula (6) in Theorem 1.4 for 2×2 matrices.

(5) Prove formula (7) in Theorem 1.4 for the integer pairs $(n, m) = (1, 4)$, $(2, 3)$.

(6) Prove formula (7) in Theorem 1.4 by induction on m.