Problem Set 1: Modeling Integer Programming Problems

(1) Suppose that you are interested in choosing to invest in one or more of 10 investment opportunities. Use 0-1 variables to model the following linear constraints.
  (a) You cannot invest in all opportunities.
  (b) You must choose at least one investment.
  (c) Investment 1 cannot be chosen if investment 3 in chosen.
  (d) Investment 4 can be chosen only if investment 2 is also chosen.
  (e) You must choose either both investments 1 and 5 or neither.
  (f) You must choose either at least one of the investments 1,2, 3 or at least two of the investments 2,4,5,6 (the "or" in this sentence is an inclusive "or" rather than a disjunctive "or").
  (g) You must choose either at least one of the investments 1,2, 3 or at least two of the investments 4,5,6 , but not both.

(2) Formulate the following problems as linear programs.
  (a) $\max\left\{\sum_{k=1}^{n-1} \min\{x_k, x_{k+1}\} \mid 0 \leq Ax \leq b\right\}$., where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
  (b) $\min\left\{\sum_{j=1}^{m} \left|a_j^T x - b_j\right| \mid 0 \leq x\right\}$, where $a_j \in \mathbb{R}^n$, and $b_j \in \mathbb{R}$ for $j = 1, 2 \ldots, m$.

(3) John is attending a Summer school where he must take four courses. There are m courses. All courses meet every day and they all last an hour. Because of the demand some courses have several sections offered at different times during the day. Section i of course j denoted (i, j) meets at time t_{ij}. All courses start on the hour for each hour beginning at 8am with the last class starting at 4pm. John's preference for a particular section of a course is influenced by the reputation of the instructor and the time of day that the section is offered. Let p_{ij} denote John's preference for section (i, j). Unfortunately, due to conflicts, John cannot always choose his preferred section.
  (a) Formulate an integer program that chooses a feasible course schedule that maximizes John's preferences.
  (b) Modify the formulation so that John has at least a one hour break between all of his classes.
  (c) Modify the formulation so that John finishes school as early in the day as possible.

(4) The LED Company must draw up a production schedule for the next 9 weeks. Jobs last several weeks and once started must continue without interruption until completion. During each week a certain number of skilled workers are required to work full time on the job. Thus, if job i lasts p_i weeks, $l_{i,j}$ workers are required in week j for $j = 1, 2, \ldots, p_i$ while the job is in progress. The total number of workers available in week t is L_t, $t = 1, 2, \ldots, 9$. Typical job data are as follows:

Job	p_i	l_{i1}	l_{i2}	l_{i3}	l_{i4}
1	3	2	3	1	-
2	2	4	5	-	-
3	4	2	4	1	5
4	4	3	4	2	2
5	3	9	2	3	-

Assume that the are n jobs to be completed.

(a) Formulate the problem of finding a feasible schedule as an ILP.

(b) Formulate the problem when the objective is to minimize the maximum number of workers required during any one of the nine weeks.

(c) Write a constraint for the situation where job 1 must start at least two weeks before job 3.

(d) Write a constraint for the situation where jobs 1 and 2 require the same machine and so cannot have there schedules overlap.

(5) A set of n jobs must be carried out on a single machine that can only do one job at a time. Each job j takes p_j hours to complete, but each job need not be run in consecutive hours till completion. However, if a job is started at the beginning of an hour, it will run in whole hour units of time. Each job also comes with a priority weight w_j where the greater the value of w_j the higher its priority. Formulate the problem of determining the sequence of jobs that maximizes the total sum of weighted priorities per hour.

(6) Quick-Car Auto Rentals faces the problem of assigning vehicles to meet the weekend demand. Quick-Car distinguishes vehicles by type: 3 compact cars (C), 5 mid-size cars (M), and 2 full-size cars (F). The rental rates depend on how many days the contract covers. Prices for compact cars are shown below. Mid-size cars carry a 10% premium and full-size cars a 20% premium.

Days	1	2	3	4
Rate ($)	40	75	100	120

The weekend customer rental requests that have been logged so far appear in the table below.

Days	**C**	**M**	**F**
Fri.–Mon.	0	1	0
Fri.–Sat.	1	2	1
Fri.-Sun.	2	2	1
Sat.–Sun.	1	3	0
Sat.–Mon.	3	0	0
Sun.–Sun.	0	1	1

(a) If Quick-Car prohibits a customer who orders one size car from renting another size, formulate an optimization problem for determining the vehicle allocation that generates the maximum revenue from these requests.

(b) If Quick-car is willing to substitute a large size vehicle for any order but with no change in price, formulate an optimization problem that maximizes the revenue that can be generated from the list of orders.

(7) Acme Shoes manufactures a line of inexpensive shoes in Pontiac. Their plant and distributes them to 5 main distribution centers (Milwaukee, Dayton, Cincinnati, Buffalo, and Atlanta) from which the shoes are shipped to retail shoe stores. The Pontiac plant can now produce $40,000$ pairs of shoes per week. The costs associated with the distribution of the shoes include freight, handling, and warehousing costs. To meet increased demand, the company has decided to build at least one new plant with a capacity of $40,000$ pairs of shoes per week. Surveys have narrowed the choice to three locations: Cincinnati, Dayton, and Atlanta. As expected, production costs would be low in Atlanta, but the distribution costs are relatively high compared to the other two locations. The data for this problem are shown below.

Current demand as well as the weekly production and fixed costs for the new plant appear in the table below.

	Buffalo	Milwaukee	Cincinnati	Dayton	Atlanta
Demand	19,000	10,000	16,000	15,000	12,000
Prod. cost/pair ($)	-	-	2.64	2.69	2.62
Fixed cost/week ($)	-	-	4,000	6,000	7,000

Distribution costs per pair ($).

To\From	Pontiac	Cincinnati	Dayton	Atlanta
Milwaukee	0.42	0.46	0.44	0.48
Dayton	0.36	0.37	0.30	0.45
Cincinnati	0.41	0.30	0.47	0.43
Buffalo	0.39	0.42	0.38	0.46
Atlanta	0.50	0.43	0.45	0.27

Formulate an optimization problem that determines where the new plant should be built to minimize the weekly production, transportation, and fixed costs, and what the associated optimal shipping schedule should be from both the Pontiac plant and the new plant.

(8) A painting operation is scheduled in blocks, and each block involves painting products with a particular color. Cleaning time is required between each pair of blocks so that the equipment can be prepared for the new color. In each cycle there is one block for each color, and the order volume determines the total painting time. In particular, the schedule length is determined by the sequence in which the blocks are scheduled because the cleaning time depends on the color in the previous block and the color in the next block. There are 6 colors. The following table gives the number of minutes required for cleaning depending on the color pair (From\To).

From\To	1	2	3	4	5	6
1	-	4	8	6	8	2
2	5	-	7	11	13	4
3	11	6	-	8	4	3
4	5	7	2	-	2	5
5	10	9	7	5	-	2
6	8	4	3	6	5	-

Formulate an optimization problem that finds the block sequence that minimizes the total amount of time spent cleaning during a full cycle.

(9) A sudden increase in the demand for smoke detectors has left Acme Alarms with insufficient capacity to meet demand. The company has seen its monthly demand from its retailers for hard wired and battery operated detectors rise to $20,000$ and $10,000$ units, respectively, and Acme wishes to continue meetng demand. Acme's production process involves three departments: fabrication, assembly, and shipping. The relevant data on production and prices are summarized in the table below.

Department	Monthly hours available	hours/unit (hard-wired)	hours/unit (battery)
fabrication	2000	0.15	0.10
assembly	4200	0.20	0.20
shipping	2500	0.10	0.15
Variable			
cost/unit ($)		18.80	16.00
retail price ($)		29.50	28.00

The Company also has the option to obtain additional units from a subcontractor, who has offered to supply up to $20,000$ units per month in any combination of hard-wired or battery oprated models at a charge of $21.50 per unit. For this price, the subcontractor will test and ship its models directly to the retailers without using Acme's production process.

Acme wants an implementable schedule, so all quantities must be integers. Formulate an integer linear program to maximize profit while meeting demand.

(10) King County is examining 4 landfill sites as candidates for the county's solid waste disposal network. The monthly costs per ton of solid waste disposal have been estimated for operating at each site and for transportation to each site from the various collection areas (tranfer stations). In addition, the amortized monthly cost for the facility at each proposed site has also been estimated. All of this data is shown in the table below. Model the problem of site selection as a mixed integer LP.

Transfer Costs/Ton ($)

To Landfill Site S_i

From\To	S_1	S_2	S_3	S_4	Tons/Month
A ($)	16	14	10	8	5000
B ($)	12	11	12	14	7000
C ($)	13	8	9	11	15000
D ($)	10	15	14	12	10000
E ($)	8	12	10	11	18000
F ($)	11	10	8	9	12000
Operating Cost/Ton	8	10	9	11	
Fixed cost/month	1000	800	700	900	

(11) A truck driver is doing a cross-country run from NY to LA and wishes to load his truck for the run. There are 5 categories of items available from a local distributor that he can load onto his truck. The items in each category have the same weight, volume, and payoff upon delivery. These are given in the table below. The truck can only hold 112 weight units and 109 volume units. Formulate a linear program that selects the items to put in the truck to maximizes the truckers payoff.

category	weight	volume	payoff
1	5	1	4
2	8	8	7
3	3	6	6
4	2	5	5
5	7	4	4