# Computing Dual LPs without Conversion to Standard Form

1. Compute the dual LP to each of the following LPs without first converting to standard form.

   (a)

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \leq 8 \\ & 0 \leq x_2 \leq 10 \end{array}$$

   (b)

$$\begin{array}{llllll} \text{maximize} & & 42x_2 & -30x_3 & & \\ \text{subject to} & x_1 & -x_2 & +x_3 & -x_4 & = 0 \\ & x_1 & & +x_3 & -x_4 & \leq 5 \\ & & 5x_2 & +x_3 & -5x_4 & = -1 \\ & 0 & \leq x_1, & & 0 \leq & x_3 \leq 20 \end{array}$$

2. Consider the mini–max problem

$$\min_{x \in \mathbb{R}^n} \max_{i=1,2,\ldots,m} \{a_i^T x - b_i\}$$

   where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$.

   (a) Show that this mini-max problem is in some sense *equivalent* to the LP

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & Ax - b \leq x_0 e, \end{array} \tag{1}$$

   where $A = (a_{ij})_{m \times n}$, $b = [b_1, b_2, \ldots, b_m]^T$, and $e \in \mathbb{R}^m$ is the vector of all ones.

   (b) Show that the dual of the LP (1) is

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y = 0, \ e^T y = 1, \\ & 0 \leq y \end{array}$$

3. Consider the system of linear inequalities and equations

$$Ax \leq b, \qquad Bx = d, \tag{2}$$

   where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{s \times t}$, $d \in \mathbb{R}^s$, and $b \in \mathbb{R}^n$. We are interested in studying the consistency of this system, that is, we are interested in determining conditions under which the solution set $S = \{x : Ax \leq b, \ Bx = d\}$ is non–empty. For this purpose, we make use of the following linear program:

$$\begin{array}{lll} \mathcal{P}: & \text{maximize} & -e^T z \\ & Ax - z & \leq b \\ & Bx & = d \\ & 0 & \leq z \end{array}$$

   where $e \in \mathbb{R}^m$ is the vector of all ones ($e = (1, 1, 1, \ldots, 1)^T$).

(a) Show that the system (2) is consistent (i.e. $S \neq \emptyset$) if and only if the optimal value in $\mathcal{P}$ is zero.

(b) Show that the dual to the LP $\mathcal{P}$ is the LP

$$\begin{array}{lll} \mathcal{D}: & \text{minimize} & b^T u + d^T v \\ & & A^T u + B^T v = 0 \\ & & 0 \leq u \leq e. \end{array}$$

(c) Show that the system $Ax \leq b$ is inconsistent (i.e. $S = \emptyset$) if and only if there are vectors $u \in I\!R^m$ and $v \in I\!R^s$ such that $0 \leq u$, $A^T u + B^T v = 0$, and $b^T u + d^T v < 0$.