Sample Theory Questions for the 408 Final

I'm still working on this list of questions so more will be added at a later date. But I though it important to get them to you early so that you can get started on them.

(1) State and prove the *Weak Duality Theorem* for linear programming.

(2) Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{k\times n}$ and $c \in \mathbb{R}^n$. Use linear programming duality to show that either

(A) there exists $x \in \mathbb{R}^n$ such that $Ax \le 0$, $Bx = 0$, and $c^T x < 0$

or

(B) there exists $u \in \mathbb{R}^m$, $v \in \mathbb{R}^k$, such that $A^T u + B^T v = -c$ and $0 \le u$,

but not both (A) and (B).

(3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Show that for any $x, d \in \mathbb{R}^n$,

$$f'(x;d) = \lim_{\substack{s\to d \\ t\downarrow 0}} \frac{f(x+ts) - f(x)}{t} .$$

(4) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. If $\bar{x}$ solves $\min_{x\in\mathbb{R}^n} f(x)$, show that $\nabla f(\bar{x}) = 0$.

(5) Given $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$, show that every feasible direction for Ω at x is also a tangent direction to Ω at x.

(6) Let $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots m$ be continuously differentiable and consider the set

$$\Omega = \{x \mid f_i(x) \le 0,\ i = 1, \dots, s,\ f_i(x) = 0,\ i = s+1, \dots, m\} .$$

Show that for every $x \in \Omega$ it is always the case that

$$T_\Omega(x) \subset \left\{d \,\middle|\, \nabla f_i(x)^T d \le 0,\ i \in I(x),\ \nabla f_i(x)^T d = 0,\ i = s+1, \dots, m\right\},$$

where $I(x) = \{i \mid i \in \{1, \dots, s\},\ f_i(x) = 0\}$.

(7) Let $C_i \subset \mathbb{R}^n$, $i = 1, \dots, N$ be convex sets. Show that the set $C = \bigcap_{j=1}^N C_j$ is also a convex set.

(8) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function, show that for any $\alpha \in \mathbb{R}$ the set $\{x \mid f(x) \le \alpha\}$ is a convex set.

(9) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing convex function. Show that the function $h : \mathbb{R}^n \to \mathbb{R}$ given by $h(x) = \phi(f(x))$ is also a convex function.

(10) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function, $A \in \mathbb{R}^{n\times k}$, and $b \in \mathbb{R}^n$. Define $h : \mathbb{R}^k \to \mathbb{R}$ by $h(y) = f(Ay + b)$. Show that h is also a convex function.

(11) Let $\ell \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ and consider the *box*

$$B = \{x \in \mathbb{R}^n \,:\, \ell_i \le x_i \le u_i, \text{ for } i = 1, 2, \dots, n\}.$$

(a) Show that the vector $d \in \mathbb{R}^n$ is an element of the tangent cone to B at the point $x \in B$ if and only if

$$\begin{aligned} d_i &\ge 0 \text{ if } x_i = \ell_i, \\ d_i &\le 0 \text{ if } x_i = u_i, \text{ and} \\ d_i & \quad \text{is free to be any real number if } l_i < x_i < u_i . \end{aligned}$$

(b) Let $f : I\!\!R^n \to I\!\!R$ be continuously differentiable. Show that if $\bar{x} \in B$ solves the problem $\min\{f(x) : x \in B\}$, then

$$\begin{array}{rcll} (\nabla f(\bar{x}))_i & \geq & 0 & \text{if } x_i = \ell_i, \\ (\nabla f(\bar{x}))_i & \leq & 0 & \text{if } x_i = u_i, \\ (\nabla f(\bar{x}))_i & = & 0 & \text{otherwise.} \end{array}$$

(12) Let $f : I\!\!R^n \to I\!\!R$ be convex function, and let $\Omega \subset I\!\!R^n$ be a convex set. If $\bar{x}$ is a local solution to the problem $\min_{x \in \Omega} f(x)$, show that $\bar{x}$ is a global solution to this problem.

(13) Let $f : I\!\!R^n \to I\!\!R$ be a differentiable convex function, and let $\Omega \subset I\!\!R^n$ be a convex set. Show that $\bar{x}$ solves the problem $\min_{x \in \Omega} f(x)$ if and only if $\nabla f(\bar{x})^T(x - \bar{x}) \geq 0$ for all $x \in \Omega$. (Hint: Use the the subdifferential inequality.)

(14) Let $f_i : I\!\!R^n \to I\!\!R, i = 0, 1, \ldots s$ be convex differentiable functions and let $f_i : I\!\!R^n \to I\!\!R, i = s + 1, \ldots, m$ be affine (i.e. $f_i(x) = a_i^T x + \alpha_i,\ i = s + 1, \ldots, m$). and consider the convex set

$$\Omega = \{x \mid f_i(x) \leq 0,\ i = 1, \ldots, s,\ f_i(x) = 0,\ i = s + 1, \ldots, m\}.$$

Show that if $\bar{x} \in I\!\!R^n$ is a KKT point for the convex program $\min_{x \in \Omega} f_0(x)$, then $\bar{x}$ must be a global solution to this problem.

(15) Let $0 \leq \delta$, $Q \in I\!\!R^{n \times n}$ be symmetric and positive definite, and $m \in I\!\!R^n$ be such that each component of m is positive and the vectors m and e are linearly independent. Define the vector $\mathrm{e} \in I\!\!R^n$ to be the vector each of whose components is the number 1. Consider the convex quadratic program

$$\begin{array}{lll} \mathcal{M} & \text{miminize} & \frac{1}{2} x^T Q x \\ & \text{subject to} & \mathrm{e}^T x = 1,\ m^T x \geq \delta. \end{array}$$

Use the KKT conditions for this problem to show that either

$$\frac{m^T Q^{-1} \mathrm{e}}{\mathrm{e}^T Q^{-1} \mathrm{e}} \geq \delta$$

in which case

$$x_{\mathrm{mv}} = \frac{1}{\mathrm{e}^T Q^{-1} \mathrm{e}} Q^{-1} \mathrm{e}$$

solves $\mathcal{M}$, or the solution to $\mathcal{M}$ is given by

$$\bar{x} = \alpha Q^{-1} e + \beta Q^{-1} m\ ,$$

where α and β is the unique solution to the 2×2 system

$$\begin{bmatrix} \mathrm{e}^T Q^{-1} \mathrm{e} & \mathrm{e}^T Q^{-1} m \\ \mathrm{e}^T Q^{-1} m & m^T Q^{-1} m \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta \end{pmatrix}.$$