Sample Theory Questions for the 408 Final

I’m still working on this list of questions so more will be added at a later date. But I though it important to get them to you early so that you can get started on them.

(1) State and prove the Weak Duality Theorem for linear programming.
(2) Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{k \times n}$ and $c \in \mathbb{R}^n$. Use linear programming duality to show that either

(A) there exists $x \in \mathbb{R}^n$ such that
$$Ax \leq 0, \quad Bx = 0, \quad \text{and} \quad c^T x < 0$$

or

(B) there exists $u \in \mathbb{R}^m$, $v \in \mathbb{R}^k$, such that
$$A^T u + B^T v = -c \quad \text{and} \quad 0 \leq u,$$

but not both (A) and (B).

(3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Show that for any $x, d \in \mathbb{R}^n$, 
$$f'(x; d) = \lim_{s \to 0} \frac{f(x + ts) - f(x)}{t}.$$

(4) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. If $\bar{x}$ solves $\min_{x \in \mathbb{R}^n} f(x)$, show that $\nabla f(\bar{x}) = 0$.

(5) Given $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$, show that every feasible direction for $\Omega$ at $x$ is also a tangent direction to $\Omega$ at $x$.

(6) Let $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ be continuously differentiable and consider the set
$$\Omega = \{ x \mid f_i(x) \leq 0, \quad i = 1, \ldots, s, \quad f_i(x) = 0, \quad i = s + 1, \ldots, m \}.$$

Show that for every $x \in \Omega$ it is always the case that
$$T_\Omega(x) \subset \{ d \mid \nabla f_i(x)^T d \leq 0, \quad i \in I(x), \quad \nabla f_i(x)^T d = 0, \quad i = s + 1, \ldots, m \},$$
where $I(x) = \{ i \mid i \in \{1, \ldots, s\}, \quad f_i(x) = 0 \}$.

(7) Let $C_i \subset \mathbb{R}^n, \quad i = 1, \ldots, N$ be convex sets. Show that the set $C = \bigcap_{j=1}^N C_j$ is also a convex set.

(8) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function, show that for any $\alpha \in \mathbb{R}$ the set
$$\{ x \mid f(x) \leq \alpha \}$$
is a convex set.

(9) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing convex function. Show that the function $h : \mathbb{R}^n \to \mathbb{R}$ given by $h(x) = \phi(f(x))$ is also a convex function.

(10) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function, $A \in \mathbb{R}^{n \times k}$, and $b \in \mathbb{R}^n$. Define $h : \mathbb{R}^k \to \mathbb{R}$ by $h(y) = f(Ay + b)$. Show that $h$ is also a convex function.

(11) Let $\ell \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ and consider the box
$$B = \{ x \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i, \quad \text{for} \quad i = 1, 2, \ldots, n \}.$$

(a) Show that the vector $d \in \mathbb{R}^n$ is an element of the tangent cone to $B$ at the point $x \in B$ if and only if
$$d_i \geq 0 \quad \text{if} \quad x_i = \ell_i,$$
$$d_i \leq 0 \quad \text{if} \quad x_i = u_i,$$
and
$$d_i \quad \text{is free} \quad \text{to be any real number} \quad \text{if} \quad l_i < x_i < u_i.$$
(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Show that if $\bar{x} \in B$ solves the problem $\min \{ f(x) : x \in B \}$, then
\[
(\nabla f(\bar{x}))_i \geq 0 \quad \text{if} \quad x_i = \ell_i,
\]
\[
(\nabla f(\bar{x}))_i \leq 0 \quad \text{if} \quad x_i = u_i,
\]
\[
(\nabla f(\bar{x}))_i = 0 \quad \text{otherwise}.
\]

(12) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $\Omega \subset \mathbb{R}^n$ be a convex set. If $\bar{x}$ is a local solution to the problem $\min_{x \in \Omega} f(x)$, show that $\bar{x}$ is a global solution to this problem.

(13) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function, and let $\Omega \subset \mathbb{R}^n$ be a convex set. Show that $\bar{x}$ solves the problem $\min_{x \in \Omega} f(x)$ if and only if $\nabla f(\bar{x})^T(x - \bar{x}) \geq 0$ for all $x \in \Omega$. (Hint: Use the the subdifferential inequality.)

(14) Let $f_i : \mathbb{R}^n \to \mathbb{R}, i = 0, 1, \ldots, s$ be convex differentiable functions and let $f_i : \mathbb{R}^n \to \mathbb{R}, i = s + 1, \ldots, m$ be affine (i.e. $f_i(x) = a_i^T x + \alpha_i, i = s + 1, \ldots, m$), and consider the convex set
\[
\Omega = \{ x \mid f_i(x) \leq 0, i = 0, 1, \ldots, s; f_i(x) = 0, i = s+1, \ldots, m \}.
\]
Show that if $\bar{x} \in \mathbb{R}^n$ is a KKT point for the convex program $\min_{x \in \Omega} f_0(x)$, then $\bar{x}$ must be a global solution to this problem.

(15) Let $0 \leq \delta, Q \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, and $m \in \mathbb{R}^n$ be such that each component of $m$ is positive and the vectors $m$ and $e$ are linearly independent. Define the vector $e \in \mathbb{R}^n$ to be the vector each of whose components is the number 1. Consider the convex quadratic program
\[
\mathcal{M} \quad \text{minimize} \quad \frac{1}{2} x^T Q x
\]
subject to
\[
e^T x = 1, \quad m^T x \geq \delta.
\]
Use the KKT conditions for this problem to show that either
\[
\frac{m^T Q^{-1} e}{e^T Q^{-1} e} \geq \delta
\]
in which case
\[
x_{mv} = \frac{1}{e^T Q^{-1} e} Q^{-1} e
\]
solves $\mathcal{M}$, or the solution to $\mathcal{M}$ is given by
\[
\bar{x} = \alpha Q^{-1} e + \beta Q^{-1} m,
\]
where $\alpha$ and $\beta$ is the unique solution to the $2 \times 2$ system
\[
\begin{bmatrix}
e^T Q^{-1} e & e^T Q^{-1} m \\
e^T Q^{-1} m & m^T Q^{-1} m
\end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta \end{pmatrix}.
\]