Solutions to Theory Quiz

(1) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be convex function, show that for any \( \alpha \in \mathbb{R} \) the set \( \{ x \mid f(x) \leq \alpha \} \) is a convex set.

**Solution:**
Let \( \alpha \in \mathbb{R} \), \( x, y \in \mathbb{R}^n \), and \( 0 \leq \lambda \leq 1 \). Then
\[
  f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) \\
  \leq (1 - \lambda)\alpha + \lambda \alpha \\
  = \alpha.
\]
Therefore, \( (1 - \lambda)x + \lambda y \in \{ x \mid f(x) \leq \alpha \} \) which shows that this set is convex.

(2) Let \( f_i : \mathbb{R}^n \to \mathbb{R}, i = 0, 1, \ldots, s \) be convex differentiable functions and let \( f_i : \mathbb{R}^n \to \mathbb{R}, i = s + 1, \ldots, m \) be affine (i.e. \( f_i(x) = a_i^T x + \alpha_i, \ i = s + 1, \ldots, m \)), and consider the convex set
\[
\Omega = \{ x \mid f_i(x) \leq 0, \ i = 1, \ldots, s; \ f_i(x) = 0, \ i = s + 1, \ldots, m \}.
\]
Show that if \( \bar{x} \in \mathbb{R}^n \) is a KKT point for the convex program \( \min_{x \in \Omega} f_0(x) \), then \( \bar{x} \) must be a global solution to this problem.

**Solution:**
Since \( \bar{x} \) is a KKT point there exists \( \bar{y} \in \mathbb{R}^m \) such that \( (\bar{x}, \bar{y}) \) is a KKT pair for \( P \). Consider the function \( h : \mathbb{R}^n \to \mathbb{R} \) given by
\[
h(x) = L(x, \bar{y}) = f_0(x) + \sum_{i=1}^{m} \bar{y}_i f_i(x).
\]
By construction, the function \( h \) is convex with \( 0 = \nabla h(\bar{x}) = \nabla L(\bar{x}, \bar{y}) \). Therefore, \( \bar{x} \) is a global solution to the problem \( \min_{x \in \Omega} h(x) \). Also note that for every \( x \in \Omega \) we have
\[
\sum_{i=1}^{m} \bar{y}_i f_i(x) \leq 0,
\]
since \( \bar{y}_i f_i(x) \leq 0 \ i = 1, \ldots, s \) and \( \bar{y}_i f_i(x) = 0 \ i = s + 1, \ldots, m \). Consequently,
\[
f_0(\bar{x}) = h(\bar{x}) \leq h(x) = L(x, \bar{y}) \\
= f_0(x) + \sum_{i=1}^{m} \bar{y}_i f_i(x) \\
\leq f_0(x)
\]
for all \( x \in \Omega \). This establishes the result.

(3) Let \( 0 \leq \delta, Q \in \mathbb{R}^{n \times n} \) be symmetric and positive definite, and \( m \in \mathbb{R}^n \) be such that each component of \( m \) is positive and the vectors \( m \) and \( e \) are linearly independent. Define the vector \( e \in \mathbb{R}^n \) to be the vector each of whose components is the number 1. Consider the convex quadratic program
\[
\mathcal{M} \quad \minimize \quad \frac{1}{2} x^T Q x \\
\text{subject to} \quad e^T x = 1, \ m^T x \geq \delta.
\]
Use the KKT conditions for this problem to show that either
\[
\frac{m^T Q^{-1} e}{e^T Q^{-1} e} \geq \delta
\]
in which case
\[
x_{mv} = \frac{1}{e^T Q^{-1} e} Q^{-1} e
\]
solves \( M \), or the solution to \( M \) is given by
\[
\bar{x} = \alpha Q^{-1} e + \beta Q^{-1} m ,
\]
where \( \alpha \) and \( \beta \) is the unique solution to the \( 2 \times 2 \) system
\[
\begin{bmatrix}
  e^T Q^{-1} e & e^T Q^{-1} m \\
  e^T Q^{-1} m & m^T Q^{-1} m
\end{bmatrix}
\begin{bmatrix}
  \alpha \\
  \beta
\end{bmatrix}
= \begin{bmatrix}
  1 \\
  \delta
\end{bmatrix}.
\]

Solution:
The KKT conditions for this quadratic program are
\begin{align*}
0 &= \Sigma w - \lambda m - \gamma e \\
\mu_b &\leq m^T w, \ e^T w = 1, \ 0 \leq \lambda \\
\lambda^T (m^T w - \mu_b) &= 0
\end{align*}
for some \( \lambda, \gamma \in \mathbb{R} \). Since the covariance matrix is symmetric and positive definite, we know that if \((w, \lambda, \gamma)\) is any triple satisfying the KKT conditions then \( w \) is necessarily a solution to \( M \). We consider two cases.

- \( \mu_b < m^T \bar{w} \): In this case, the complementarity condition (3) implies \( \lambda = 0 \). Hence the KKT conditions reduce to the two equations \( 0 = \Sigma \bar{w} - \gamma e \) and \( e^T \bar{w} = 1 \). Multiplying the first through by \( \Sigma^{-1} \) yields \( \bar{w} = \gamma \Sigma^{-1} e \). Multiplying this equation through by \( e \) and using the fact that \( e^T \bar{w} = 1 \) gives \( \gamma = (e^T \Sigma^{-1} e)^{-1} \). Therefore,
\[
\bar{w} = (e^T \Sigma^{-1} e)^{-1} \Sigma^{-1} e.
\]

It is important to note that this value of \( w \) gives the smallest possible variance over all portfolios since it solves the problem
\[
\mathcal{M}_{\text{min-var}} : \text{minimize} \quad \frac{1}{2} w^T \Sigma w \\
\text{subject to} \quad e^T w = 1 .
\]

Consequently, the return associated with the least variance solution is
\[
\mu_{\text{min-var}} = \frac{m^T \Sigma^{-1} e}{e^T \Sigma^{-1} e} .
\]

We denote the set of weights associated with the minimum variance solution \( \bar{w} \) by \( w_{\text{min-var}} \) as well.

Finally observe that is the minimum variance weights \( w_{\text{min-var}} \) are feasible for \( M \), that is, if \( m^T w_{\text{min-var}} \geq \mu_b \), then \( w_{\text{min-var}} \) must be the solution to \( M \) since it the solution to the problem \( \mathcal{M}_{\text{min-var}} \). Therefore, when solving \( M \) one first computes \( w_{\text{min-var}} \) and checks to see if the inequality \( m^T w_{\text{min-var}} \geq \mu_b \) holds. If it does hold, then \( w_{\text{min-var}} \) solves \( M \) and no further work is required. If it does not hold then you know that the constraint \( m^T w = \mu_b \) at the solution to \( M \).
• $\mu_u = m^T \omega$: Multiplying (1) through by $\Sigma^{-1}$ gives

$$\omega = \lambda \Sigma^{-1} m + \gamma \Sigma^{-1} e.$$  

(4)

Using this formula for $\omega$ and (2), we get the two equations

$$\mu_u = \lambda m^T \Sigma^{-1} m + \gamma m^T \Sigma^{-1} e$$
$$1 = \lambda m^T \Sigma^{-1} e + \gamma e^T \Sigma^{-1} e,$$

or equivalently, the $2 \times 2$ matrix equation

$$\begin{bmatrix} m^T \Sigma^{-1} m & m^T \Sigma^{-1} e \\ m^T \Sigma^{-1} e & e^T \Sigma^{-1} e \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} \mu_u \\ 1 \end{bmatrix}.$$  

(5)

This concludes the proof.