1. (a) Let $x \in \mathbb{R}$. Show that

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{(n+1)}}{1 - x} \ .$$

Solution

$$\begin{array}{rcll} S_n & = & 1 & +x + x^2 + \ldots + x^n \\ xS_n & = & & +x + x^2 + \ldots + x^n + x^{n+1} \\ S_n - xS_n & = & 1 & -x^{n+1} = (1-x)S_n \end{array}$$

Therefore, $S_n = \frac{1-x^{n+1}}{1-x}$.

(b) A lottery advertises that it pays the winner $\$1,000,000$. However, this prize money is paid in 20 anual installments of $\$50,000$ each with the first installment paid on the date that the winning ticket is cashed in. At a prevailing interest rate of 5%, write a numerical expression for the present value of the lottery prize on the date that it is cashed in. You must use Part (a) above to reduce the complexity of this numerical expression.

Solution

$$\begin{aligned} PV &= 50,000 + \frac{50,000}{1.05} + \ldots + \frac{50,000}{(1.05)^{19}} \\ &= 50,000 \sum_{k=0}^{19} \left(\frac{1}{1.05}\right)^k \\ &= 50,000 \frac{1 - \left(\frac{1}{1.05}\right)^{20}}{1 - \frac{1}{1.05}} \\ &= 50,000 \frac{1.05}{.05}\left(1 - \left(\frac{1}{1.05}\right)^{20}\right) \\ &= 1,050,000\left(1 - \left(\frac{1}{1.05}\right)^{20}\right) \end{aligned}$$

2. XXX Corporation wishes to issue a callable bond at par where current yields are 5% (that is, XXX Corp. wishes to issue a 5% callable bond). To call the bond, XXX Corp. must pay a premium. The terms of the bond require that the premium needs to be declared at the outset. XXX Corp. determines that it should have the option to call the bond after year 3 if the yield on this type of bond rises by more that 2% in that time. Derive a formula for the premium that justifies this calling strategy where the premium is to be determined as a multiple of the face value of the bond.

Solution

$$\begin{aligned}
F &= \text{Face value of the bond} \\
n &= \text{years to maturity} \\
\text{Premium} &= xF \quad \text{for some } x > 0. \\
xF &= \frac{.05F}{(1.07)} + \frac{.05F}{(1.07)^2} + \ldots + \frac{1.05F}{(1.07)^{(n-3)}} \\
&= F\left[\frac{1}{(1.07)^{(n-3)}} + \frac{.05}{1.07}\sum_{k=0}^{n-4}\left(\frac{1}{1.07}\right)^k\right] \\
&= F\left[\frac{1}{(1.07)^{(n-3)}} + \frac{.05}{1.07}\left(\frac{1-\left(\frac{1}{1.07}\right)^{n-3}}{1-\frac{1}{1.07}}\right)\right] \\
&= F\left[\frac{1}{(1.07)^{(n-3)}} + \frac{.05}{1.07}\frac{1.07}{.07}\left(1-\left(\frac{1}{1.07}\right)^{n-3}\right)\right] \\
&= F\left[\frac{1}{(1.07)^{(n-3)}} + \frac{535}{735}\left(1-\left(\frac{1}{1.07}\right)^{n-3}\right)\right].
\end{aligned}$$

Therefore,

$$x = \frac{1}{(1.07)^{(n-3)}} + \frac{535}{735}\left(1-\left(\frac{1}{1.07}\right)^{n-3}\right).$$

3. A municipality has the following schedule of liabilities with the first payment occuring one year from now:

Year	1	2	3	4	5	6	7	8
Dollars	12,000	18,000	20,000	20,000	16,000	15,000	12,000	10,000

The bonds available for purchase today are given in the following table. All bonds have a face value of \$100. The coupon figure is anual. For example, bond 5 costs \$98 today and pays back \$4 in year 1, \$4 in year 2, \$4 year 3, and \$104 in year 4. All of these bonds are widely available and can be purchased in large quantities at the stated prices.

Bond	1	2	3	4	5	6	7	8	9	10
Price	102	99	101	98	98	104	100	101	102	104
Coupon	5%	4%	5%	4%	4%	5%	3%	4%	5%	5%
Maturity	1	2	2	3	4	5	5	6	7	8

Formulate a linear program to find the least cost portfolio of bonds to purchase today to meet the obligations of the municipality over the next 8 years. Assume that any surplus cash in a given year can be re-invested for a year at annual rate of 6%.

Solution

Decision Variables

$$\begin{aligned} x_i &= \text{number of bonds } i \text{ purchased } i = 1, 2, \ldots, 10. \\ s_j &= \text{dollars in savings in year } j = 0, 1, \ldots, 8. \end{aligned}$$

Objective

$$\min c^T x + s_0,$$

where $c = (102, 99, 101, 98, 98, 104, 100, 101, 102, 104)^T$.

Constraints

$$\begin{aligned}
105x_1 + 4x_2 + 5x_3 + 4x_4 + 4x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} + (1.06)s_0 &= 12,000 + s_1 \\
104x_2 + 105x_3 + 4x_4 + 4x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} + (1.06)s_1 &= 18,000 + s_2 \\
104x_4 + 4x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} + (1.06)s_2 &= 20,000 + s_3 \\
104x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} + (1.06)s_3 &= 20,000 + s_4 \\
105x_6 + 103x_7 + 4x_8 + 5x_9 + 5x_{10} + (1.06)s_4 &= 16,000 + s_5 \\
104x_8 + 5x_9 + 5x_{10} + (1.06)s_5 &= 15,000 + s_6 \\
105x_9 + 5x_{10} + (1.06)s_6 &= 12,000 + s_7 \\
105x_{10} + (1.06)s_7 &= 10,000 + s_8 \\
0 \leq x_i, s_j \; i = 1, \ldots, 10, \; j = 0, \ldots, 8 &
\end{aligned}$$

4. (a) Let $f : I\!R^n \to I\!R$ be a convex function and let $\phi : I\!R \to I\!R$ be a non-decreasing convex function. Show that the function $g : I\!R^n \to I\!R$ defined by $g(x) = \phi(f(x))$ is also a convex function.

Solution
Let $0 \le \lambda \le 1$ and $x, y \in I\!R^n$. Since f is convex,

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y).$$

Hence,

$$\phi(f((1-\lambda)x + \lambda y)) \le \phi((1-\lambda)f(x) + \lambda f(y))$$

since ϕ is non-decreasing. But then

$$\begin{aligned} g((1-\lambda)x + \lambda y) &= \phi(f((1-\lambda)x + \lambda y)) \\ &\le \phi((1-\lambda)f(x) + \lambda f(y)) \\ &\le (1-\lambda)\phi(f(x)) + \lambda\phi(f(y)) \\ &= (1-\lambda)g(x) + \lambda g(y), \end{aligned}$$

where the second inequality follows from the convexity of ϕ. Therefore, g is convex.

(b) Show that the function $f(x) = \mathrm{e}^{\|x\|}$ is a convex function.

Solution
Let $0 \le \lambda \le 1$ and $x, y \in I\!R^n$. Set $h(x) = \|x\|$. Then h is convex:

$$\|(1-\lambda)x + \lambda y\| \le \|(1-\lambda)x\| + \|\lambda y\| = (1-\lambda)\|x\| + \lambda\|y\|.$$

Let $\phi(y) = \mathrm{e}^y$. The ϕ is non-decreasing since $\phi'(y) = \mathrm{e}^y \ge 0 \;\forall\, y$, and ϕ is convex since $\phi''(y) = \mathrm{e}^y \ge 0 \;\forall\, y$. Therefore, $f(x) = \mathrm{e}^{\|x\|} = \phi(h(x))$ is convex by Part (a) of this problem.