Description of the Final

The final exam will have 5 sections. Each section will be worth 60 points for a total of 300 points. The following titles describe the content of each section.

(1) Fixed Income Securities: Bonds
   (a) Cash flow streams
   (b) present value
   (c) internal rate of return
   (d) the bond pricing formula
(2) Linear Programming: modeling and duality theory
   (a) Transformation to standard form
   (b) The Weak and Strong duality theorems
   (c) Computation of general LP duals
   (d) Graphical solution of 2 dimensional LPs
   (e) Modeling LPs
   (f) Cash flow matching LPs
(3) Nonlinear Programming: Constrained and Unconstrained
   (a) Verification of optimality conditions for unconstrained problems (1st- and 2nd-order conditions)
   (b) Computation of the tangent cone and cone of feasible directions
   (c) Verification of regularity
   (d) Location of KKT points
   (e) Verification of optimality conditions for constrained problems (1st- and 2nd-order conditions)
(4) Convexity
   (a) Testing for and verification of the convexity of a set and a function
   (b) Constrained and Unconstrained optimality conditions with convexity
(5) The Markowitz QP
   (a) Formulate and solve Markowitz QPs
   (b) Interpret the solution of Markowitz QPs
   (c) minimum variance and market weights

MATH 408 SAMPLE FINAL EXAM **NAME (Please print):** ______________________

March 11, 2004

The exam has 8 pages (including this cover page) with 5 question areas. Please count the pages now to make sure that you have a complete exam. If your exam is not complete, you must request a new exam. You may use two pages of notes as resources during the exam. Each question area is worth 60 points for a total of 300 points. Show all of your work and follow the directions provided. Partial credit will be given for partial solutions.

Problem	Score
1	
2	
3	
4	
5	
Total	

1. *Fixed Income Securities: Bonds*

Fifteen years ago today Mary purchased a 10% bond with 30 years to maturity and a face value of \$10,000. Assume the yield at the time of purchase was 10% and the yield today is 3%. Also assume that the coupon payments occur only once a year.

I expect you to compute a numerical value for each question. Since calculators are not allowed, you may use the following helpful numbers:

$$\left(\frac{1}{1.1}\right)^{30} = 0.057, \quad \left(\frac{1}{1.1}\right)^{15} = 0.24, \quad \left(\frac{1}{1.03}\right)^{30} = 0.41, \quad \left(\frac{1}{1.03}\right)^{15} = 0.64$$

(a)(20 points) At what price did Mary purchase the bond 15 years ago?

(b)(40 points) At what price can she sell the bond today?

2. *Linear Programming*

(a)(40 points) A municipality has the following schedule of liabilities:

Year	2004	2005	2006	2007	2008	2009	2010	2011
Dollars	12,000	18,000	20,000	20,000	16,000	15,000	12,000	10,000

The bonds available for purchase today are given in the following table. All bonds have a face value of \$100. The coupon figure is annual. For example, bond 5 costs \$98 today and pays back \$4 in 2004, \$4 in 2005, \$4 in 2006, and \$104 in 2007. All of these bonds are widely available and can be purchased in large quantities at the stated prices.

Bond	1	2	3	4	5	6	7	8	9	10
Price	102	99	101	98	98	104	100	101	102	104
Coupon	5%	4%	5%	4%	4%	5%	3%	4%	5%	5%
Maturity	2004	2005	2005	2006	2007	2008	2008	2009	2010	2011

Formulate a linear program to find the least cost portfolio of bonds to purchase today to meet the obligations of the municipality over the next 8 years. Assume that any surplus cash in a given year can be re-invested for a year at annual rate of 2.5%.

2.(b)(20 points) Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. Use the Strong Duality Theorem of linear programming to show that the system

$$Ax \leq 0 \quad \text{and} \quad c^T x > 0$$

is unsolvable if and only if the system

$$A^T y = c \quad \text{and} \quad 0 \leq y$$

is solvable.

3. *Nonlinear Programming*
Let $\ell \in I\!R^n$ and $u \in I\!R^n$ and consider the *box*

$$B = \{x \in I\!R^n \ : \ \ell_i \le x_i \le u_i, \text{ for } i = 1, 2, \ldots, n\}.$$

a)(30 points) Show that the vector $d \in I\!R^n$ is an element of the tangent cone to B at the point $x \in B$ if and only if

$$\begin{aligned} d_i &\ge 0 \text{ if } x_i = \ell_i, \\ d_i &\le 0 \text{ if } x_i = u_i, \text{ and} \\ d_i & \quad \text{is free to be any real number if } l_i < x_i < u_i \ . \end{aligned}$$

b)(30 points) Let $f : I\!R^n \to I\!R$ be continuously differentiable. Show that if $\bar{x} \in B$ solves the problem $\min\{f(x) \ : \ x \in B\}$, then

$$\begin{aligned} (\nabla f(\bar{x}))_i &\ge 0 \quad \text{if } x_i = \ell_i, \\ (\nabla f(\bar{x}))_i &\le 0 \quad \text{if } x_i = u_i, \\ (\nabla f(\bar{x}))_i &= 0 \quad \text{otherwise.} \end{aligned}$$

4. *Convexity*
(a) (30 points) Let $A \in I\!R^{m\times n}$ and $C \subset I\!R^m$. Show that the set

$$\{x \mid Ax \in C, x \in I\!R^n\}$$

is a convex subset of $I\!R^n$.

(b) (30 points) Let $\gamma \in I\!R$, $g \in I\!R^n$, and $H \in I\!R^{n\times n}$ with H symmetric, and consider the quadratic function

$$q(x) = \gamma + g^T x + \frac{1}{2} x^T H x \ .$$

Given $\bar{x} \in I\!R^n$ show that there always exists $\alpha \geq 0$ such that the function

$$g_\alpha(x) = g(x) + \frac{\alpha}{2}\|x - \bar{x}\|^2$$

is convex.
BONUS POINTS: (5 points) What is the smallest value that α can take that insures that g_α is convex?

5. *The Markowitz QP*

(a) (20 points) Consider the *no short selling* minimum variance Markowitz portfolio problem

$$\mathcal{M}_{ns} \;:\; \begin{array}{ll} \text{minimize} & \frac{1}{2}w^T\Sigma w \\ \text{subject to} & 0 \leq w,\ e^T w = 1\ , \end{array}$$

where $\Sigma \in I\!R^{n\times n}$ is symmetric and positive definite, and $e \in I\!R^n$ is the vector of all ones. Use the complementarity conditions for this problem to show that the unique solution must satisfy

$$e \leq \frac{\Sigma w}{w^T \Sigma w}.$$

5.(b)(40 points) Consider the following Markowitz mean-variance portfolio optimization problem for the two financial assets $i = 1, 2$ having rates of return r_1 and r_2:

$$E(r_1) = 0.02, \ E(r_2) = 0.01, \ E(r_3) = 0.02,$$
$$\mathrm{var}(r_1) = 0.04, \ \mathrm{var}(r_2) = 0.02, \ \mathrm{var}(r_3) = 0.02$$
$$\mathrm{cov}(r_1, r_2) = 0.02, \ \mathrm{cov}(r_1, r_3) = 0, \ \text{and } \mathrm{cov}(r_2, r_3) = 0.$$

The target rate of return on the portfolio is 0.03.

(i) What is the covariance matrix Σ for r_1 and r_2 and what is its inverse?

(ii) Solve this Markowitz QP.