MATH 408 SOLUTIONS TO THE SAMPLE FINAL EXAM

1. A municipality has the following schedule of liabilities:

<table>
<thead>
<tr>
<th>Year</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dollars</td>
<td>12,000</td>
<td>18,000</td>
<td>20,000</td>
<td>20,000</td>
<td>16,000</td>
<td>15,000</td>
<td>12,000</td>
<td>10,000</td>
</tr>
</tbody>
</table>

The bonds available for purchase today are given in the following table. All bonds have a face value of $100. The coupon figure is annual. For example, bond 5 costs $98 today and pays back $4 in 2004, $4 in 2005, $4 in 2006, and $104 in 2007. All of these bonds are widely available and can be purchased in large quantities at the stated prices.

<table>
<thead>
<tr>
<th>Bond</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>102</td>
<td>99</td>
<td>101</td>
<td>98</td>
<td>98</td>
<td>104</td>
<td>100</td>
<td>101</td>
<td>102</td>
<td>104</td>
</tr>
<tr>
<td>Coupon</td>
<td>5%</td>
<td>4%</td>
<td>5%</td>
<td>4%</td>
<td>5%</td>
<td>3%</td>
<td>4%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
</tr>
</tbody>
</table>

Formulate a linear program to find the least cost portfolio of bonds to purchase today to meet the obligations of the municipality over the next 8 years. Assume that any surplus cash in a given year can be re-invested for a year at annual rate of 2.5%.

This is a standard cash flow matching problem.

Decision Variables:

\[ x_i = \text{number of bonds } i \text{ to purchase, } i = 1, 2, \ldots, 10 \]
\[ s_j = \text{surplus cash at the end of period } j, j = 1, 2, \ldots, 8 \]

Objective:

\[ \min 102x_1 + 99x_2 + 101x_3 + 98x_4 + 98x_5 + 104x_6 + 100x_7 + 101x_8 + 102x_9 + 104x_{10} \]

Constraints:

\[ 12000 = 105x_1 + 4x_2 + 5x_3 + 4x_4 + 4x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} - s_1 \]
\[ 1800 = 104x_2 + 105x_3 + 4x_4 + 4x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} + 0.025s_1 - s_2 \]
\[ 20000 = 104x_4 + 4x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} + 0.025s_2 - s_3 \]
\[ 20000 = 104x_5 + 5x_6 + 3x_7 + 4x_8 + 5x_9 + 5x_{10} + 0.025s_3 - s_4 \]
\[ 16000 = 105x_6 + 103x_7 + 4x_8 + 5x_9 + 5x_{10} + 0.025s_4 - s_5 \]
\[ 15000 = 104x_8 + 5x_9 + 5x_{10} + 0.025s_5 - s_6 \]
\[ 12000 = 105x_9 + 5x_{10} + 0.025s_6 - s_7 \]
\[ 10000 = 105x_{10} + 0.025s_7 - s_8 \]
\[ 0 \leq x_i, \ i = 1, 2, \ldots, 10 \]
\[ 0 \leq s_j, \ j = 1, 2, \ldots, 8 \]
2. Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. Use the Strong Duality Theorem of linear programming to show that the system

$$Ax \leq 0 \quad \text{and} \quad c^T x > 0$$

is unsolvable if and only if the system

$$A^T y = c \quad \text{and} \quad 0 \leq y$$

is solvable.

Consider the following LP $\mathcal{P}$ and its dual $\mathcal{D}$:

$$\mathcal{P} : \min c^T x \quad \text{subject to} \quad Ax \leq 0$$

$$\mathcal{D} : \min 0^T y \quad \text{subject to} \quad A^T y = c, \ 0 \leq y$$

Observe that if there is an $\bar{x}$ such that $A \bar{x} \leq 0$ and $c^T \bar{x} > 0$, then for all $\lambda > 0$ we have $\bar{x}_\lambda = \lambda \bar{x}$ also satisfies $Ax \leq 0$ and $c^T \bar{x}_\lambda = \lambda c^T \bar{x} > 0$. Indeed, as $\lambda \uparrow +\infty$ we have $c^T \bar{x}_\lambda \uparrow +\infty$. Therefore, there exists $x$ such that $Ax \leq 0$ and $c^T x > 0$ if and only if the optimal value in $\mathcal{P}$ is $+\infty$, or equivalently, the system $Ax \leq 0$, $c^T x > 0$ has no solution if and only if the optimal value in $\mathcal{P}$ is zero (since $x = 0$ is feasible for $\mathcal{P}$). The Strong Duality Theorem of linear programming says that the optimal solution to $\mathcal{P}$ is zero if and only if the optimal solution to $\mathcal{D}$ is zero. This proves the result, since $\mathcal{D}$ has an optimal value of zero if and only if it is feasible.

This problem is just one of many so called theorems of the alternative. Another is,

The system $Ax \leq 0$, $Ex = 0$, $c^T x > 0$ is not solvable if and only if the the system $A^T u + E^T v = c$, $0 \leq u$ is solvable.

This is proved in essentially the same manner. What are the associated primal and dual LPs to be used in this case?
3. Consider the no short selling minimum variance Markowitz portfolio problem

\[ M_{ns} : \text{minimize } \frac{1}{2} w^T \Sigma w \]
subject to \( 0 \leq w, \ e^T w = 1 \),

where \( \Sigma \in \mathbb{R}^{n \times n} \) is symmetric and positive definite, and \( e \in \mathbb{R}^n \) is the vector of all ones. Use the complementarity conditions for this problem to show that the unique solution must satisfy

\[ e \leq \frac{\Sigma w}{w^T \Sigma w}. \]

The KKT conditions for \( M_{ns} \) are

\[ e^T w = 1, \ 0 \leq w, \ 0 \leq \Sigma w + \gamma e \quad \text{and} \quad w^T (\Sigma w + \gamma e) = 0 \]

for some \( \gamma \in \mathbb{R} \). Since \( \Sigma \) is positive definite and the feasible region for \( M_{ns} \) is compact we know that a unique solution to \( M_{ns} \) exists and it is characterized by the KKT conditions. That is, any point satisfying the KKT conditions for some \( \gamma \in \mathbb{R} \) is necessarily the solution to \( M_{ns} \). The complementarity condition \( w^T (\Sigma w + \gamma e) = 0 \) and the equation \( e^T w = 1 \) implies that \( \gamma = -w^T \Sigma w \). Plugging this into the inequality \( 0 \leq \Sigma w + \gamma e \) gives \( 0 \leq \Sigma w - w^T \Sigma w e \). Since \( \Sigma \) is positive definite and \( w \neq 0 \) (as \( e^T w = 1 \)), we know that \( w^T \Sigma w > 0 \). Therefore, the inequality \( 0 \leq \Sigma w - w^T \Sigma w e \) is equivalent to the inequality

\[ e \leq \frac{\Sigma w}{w^T \Sigma w}. \]
4. Consider the following Markowitz mean-variance portfolio optimization problem for the two financial assets $i = 1, 2$ having rates of return $r_1$ and $r_2$:

$$E(r_1) = .08, \; E(r_2) = .02, \; \text{var}(r_1) = 0.04, \; \text{var}(r_2) = 0.02, \; \text{and} \; \text{cov}(r_1, r_2) = 0.02.$$  

The target rate of return on the portfolio is 0.05.

(i) What is the covariance matrix $\Sigma$ for $r_1$ and $r_2$ and what is its inverse?

$$\Sigma = \frac{1}{50} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \Sigma^{-1} = 50 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad m = \frac{1}{50} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

(ii) Solve the Markowitz QP stated on the previous page.

$$w_{\text{minvar}} = \frac{\Sigma^{-1} e}{e^T \Sigma^{-1} e} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad w_{\text{market}} = \frac{\Sigma^{-1} m}{e^T \Sigma^{-1} m} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Since $m^T w_{\text{minvar}} = .02 < .05$ we obtain the solution to the Markowitz problem by solving for $\alpha$ in the equation

$$.05 = m^T (w_{\text{minvar}} + \alpha(w_{\text{market}} - w_{\text{minvar}}))$$

which can be written as

$$5 = 2 \left( \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right)^T \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha \left( \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right) = 2 + 18\alpha$$

so that $\alpha = 1/6$ and the optimal portfolio weights are

$$w_{\text{optimal}} = w_{\text{minvar}} + \alpha(w_{\text{market}} - w_{\text{minvar}}) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$
5. Show that the function \( f : \mathbb{R}^n \to \mathbb{R} \) given by
\[
f(x) = e^{\frac{1}{2}||x||^2}
\]
is convex.

\[
\nabla f(x) = xe^{\frac{1}{2}||x||^2}
\]
\[
\nabla^2 f(x) = I e^{\frac{1}{2}||x||^2} + xx^T e^{||x||^2}
\]

Since the matrix \( I e^{\frac{1}{2}||x||^2} \) is positive definite and the matrix \( xx^T e^{||x||^2} \) is positive semi-definite, we have that \( \nabla^2 f(x) \) is everywhere positive definite and so \( f \) is convex.
6. Let $\ell \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ with $l_i < u_i$, $i = 1, 2, \ldots, n$ and consider the box

$$
B = \{ x \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i, \text{ for } i = 1, 2, \ldots, n \}.
$$

a) Show that the vector $d \in \mathbb{R}^n$ is an element of the tangent cone to $B$ at the point $x \in B$ if and only if

$$
d_i \geq 0 \text{ if } x_i = \ell_i,
$$

$$
d_i \leq 0 \text{ if } x_i = u_i, \text{ and}
$$

$$
d_i \text{ is free to be any real number if } l_i < x_i < u_i.
$$

Let $F_B(x)$ denote the set of directions satisfying the conditions listed above. Clearly, every $d \in F_B(x)$ is a feasible direction for $\Omega$ at $x$ and so $F_B(x) \subset T_\Omega(x)$. Therefore, we need only show that $T_\Omega(x) \subset F_B(x)$ to establish the result. To this end let $d \in T_\Omega(x)$. Then there exist sequences $\{x^k\} \subset \Omega$ and $\{t_k\} \subset \mathbb{R}$ with

$$
x^k \to x, \ t_k \downarrow 0, \text{ and } (x^k - x)/t_k \to d.
$$

Define $d^k = (x^k - x)/t^k$ for $k = 1, 2, \ldots,$

$$
L(x) = \{i : x_i = l_i\}, \text{ and } U(x) = \{i : x_i = u_i\}.
$$

Then, for each $i = 1, 2, \ldots, n$ we have

$$
l_i \leq x^k = x + t^k d^k \leq u_i.
$$

Consequently,

$$
x_i = l_i \leq x_i + t^k d^k_i \quad i \in L(x),
$$

$$
x_i = u_i \geq x_i + t^k d^k_i \quad i \in U(x), \text{ and}
$$

$$
l_i < x_i + t^k d^k_i < u_i \quad i \notin L(x) \cup U(x),
$$

or equivalently,

$$
0 \leq d^k_i \quad i \in L(x), \quad (1)
$$

$$
0 \geq d^k_i \quad i \in U(x), \quad (2)
$$

$$
\frac{l_i - x_i}{t_k} < d^k_i < \frac{u_i - x_i}{t_k} \quad i \notin L(x) \cup U(x) \quad (3)
$$

for all $k$ sufficiently large (since $t^k d^k_i$ gets arbitrarily small in magnitude). Since $(l_i - x_i) < 0$ and $0 < (u_i - x_i)$ for all $i \notin L(x) \cup U(x)$, we also have

$$
\frac{l_i - x_i}{t_k} \to -\infty \quad \text{and} \quad \frac{u_i - x_i}{t_k} \to +\infty.
$$

Now taking the limit in (1)-(3), we get the result.
b) Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be continuously differentiable. Show that if \( \bar{x} \in B \) solves the problem \( \min \{ f(x) : x \in B \} \), then

\[
(\nabla f(\bar{x}))_i \geq 0 \quad \text{if} \quad x_i = \ell_i,
\]
\[
(\nabla f(\bar{x}))_i \leq 0 \quad \text{if} \quad x_i = u_i,
\]
\[
(\nabla f(\bar{x}))_i = 0 \quad \text{otherwise}.
\]

The KKT conditions for this problem imply the existence of non-negative vectors \( y \in \mathbb{R}^n_+ \) and \( z \in \mathbb{R}^n_+ \) such that

\[
\nabla f(\bar{x}) = z - y \quad \text{with} \quad y^T(x - u) = 0, \quad \text{and} \quad z^T(l - x) = 0.
\]

Therefore, \( y_i = 0 \) if \( \bar{x}_i < u_i \) and \( z_i = 0 \) if \( l_i < \bar{x}_i \). Consequently, if \( \bar{x}_i = u_i > l_i \), then \( (\nabla f(\bar{x}))_i = -y_i \leq 0 \). Similarly, if \( x_i = l_i < u_i \), then \( (\nabla f(\bar{x}))_i = z_i \geq 0 \). Finally, if \( l_i < x_i < u_i \), then \( y_i = 0 = z_i \) and so \( (\nabla f(\bar{x}))_i = 0 \).